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THE LETHAL VOLUME OF A CONTINUOUS ROD WARHEAD

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THE LETHAL VOLUME OF A CONTINUOUS ROD WARHEAD

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ABSTRACT

The lethal volume of a continuous rod warhead is defined as that volume in which a target reference point must be located at burst if the target is to be subjected to a lethal cut from the rod. The lethal volume is a function of the characteristics of the target, warhead, and intercept dynamics.

A continuous rod warhead subjects a target to lethal damage by ejecting a loop of metal laterally to cut a structural member in the target. As the loop is ejected it expands in a circle until the radius of breakage is reached.

The target is assumed to be a single, linear, rigid, and non-rotating structural member. The lethal volume is obtained for this target by examining the relative motion of the target and of the rod with respect to the missile carrying the warhead. Under the assumption of constant rod velocity, the lethal volume is shown to be composed of linear segments joining two identical right elliptic cones.

Expressions for the size and probsbility content of the lethal volume are then obtained for the constant rod velocity case. A numerical scheme is discussed with which the probability content of the lethal volume can be evaluated.

Parameter variation studies, which can be performed with the lethal volume model, are next discussed. As examples, the variation in the size and probability content of the lethal volume is computed for several combinations of the input parameters.

The thesis is closed with two proposed schemes for improving the model. First, a means of improving the target model such thst

it bears a closer resemblance to an airplane is discussed. Finally, a means of accounting for variable rod velocity is given.

CHAPTER I

INTRODUCTION

A. LETHAL VOLUMES

The lethal volume of a warhead is that volume in which a target reference point must be located at the time of burst if it is to be subjected to some lethal effect from the warhead. Thus a necessary and sufficient condition for target kill is that the reference point be located in the lethal volume at the time of burst. The location of the lethal volume is usually considered relative to the center of the warhead, but is occasionally considered relative to some other point such as a point on the target. Here, the former procedure will be followed.

As an elementary example of a lethal volume, consider a warhead which destroys all point targets within a distance R of the center of the warhead at the time of burst. The lethal volume would then be a sphere of radius R centered about the warhead. The distance R is called the lethal radius of the warhead. The lethal radius is primarily a function of the warhead and target characteristics but is also influenced by environmental factors such as atmospheric density.

The usefulness of the concept of a lethal volume lies largely in its relation to the kill probability of an operable missile. A missile is said to be operable unless it fails because of weapon system malfunctions. The kill probability of an operable missile, P, is defined as the probability that, under a given set of target, missile, warhead and intercept conditions, an operable missile will result in a target kill.

If f(x,y,z,) is the probability density function describing the location of the target reference point relative to the warhead at the time of burst, then the kill probability of an operable missile is given by,

$$P = \iiint_V f(x,y,z)\,dxdydz \quad ,$$

where V is the lethal volume. If,

$$f(x,y,z) = \frac{1}{(2\pi)^{3/2}\,\sigma^3}\, e^{-1/2\sigma^2[x^2+y^2+z^2]}$$

describes the distribution of point targets about the center of the warhead for the spherical lethal volume considered earlier, then

$$P = \iiint_V \frac{1}{(2\pi)^{3/2}\,\sigma^3}\, e^{-1/2\sigma^2[x^2+y^2+z^2]}\,dxdydz \quad .$$

If $P(\nu \leq R)$ denotes the probability that $\nu \leq R$, then $P = P\left(\frac{x^2+y^2+z^2}{\sigma^2} \leq \frac{R^2}{\sigma^2}\right)$

This is simply $P\left(\chi^2_3 \leq R^2/\sigma^2\right)$

where χ^2_3 has a chi-square distribution with three degrees of freedom.[1] This probability can be found from a table of the chi-square cumulative distribution.

In practice, it is rarely possible to evaluate P in terms of tabulated functions. For this reason, numerical integration of the integral expression for P must normally be used.

Once the relationship between the lethal volume and the kill probability has been established, system parameter tradeoffs can be made. To perform tradeoffs, system parameters are relsted to the lethal radius of the warhead and the guidance accuracy of the

[1] H. W. Lilliefors, "A Hand-Computation Determination of Kill Probability for Weapons Having Spherical Lethal Volume", OPERATIONS RESEARCH JOURNAL, Vol. 5, No. 3, June 1957, pp. 416-421.

missile. These parameters can be varied and their effect on kill probability observed. If the model is sensitive to environmental factors, their effects on kill probability can also be investigated. On the other side, studying the effects of hardening the target or forcing the burst to occur closer to the target is a problem frequently investigated. In all cases significant changes in the kill probability are sought for a minimum of cost.

In addition to the kill probability, another measure of warhead effectiveness which is often used is the size of the lethal volume. Since the kill probability is obtained by integrating over the lethal volume, the variation of the size of the lethal volume can be used as a crude measure of the variation in kill probability.

The size of the lethal volume as a measure of effectiveness is useful since it is easier and cheaper to compute than the probability of kill. For instance, the IBM 7090 computer program for the lethal volume of a continuous rod warhead based on this thesis computes the size of the volume at less than one-fourth the cost of the calculation of kill probability.[2] For this reason, the size of the lethal volume is preferred as a measure of warhead effectiveness when the size of the volume adequately reflects the parameters under study.

B. CONTINUOUS ROD WARHEADS

To form a continuous rod warhead, a single long continuous metal rod is folded onto the warhead over a high explosive charge.

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[2] Vitro Laboratories Program R-16, Silver Spring, Maryland, 1964.

Upon detonation the rod expands in a circle normal to and centered at the missile axis, as shown in figure 1A. The circle of metal expands as a continuous rod until the circumference of the circle is approximately equal to the length of the unfolded rod. Then the rod breaks randomly into discrete lengths. The radius of the circle of metal at the time of breakup is called the maximum opening radius of the rod and will be denoted by R_L.

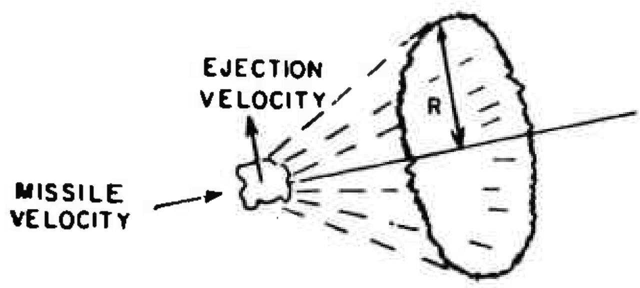
The speed, thickness, and mass of the rod are such that if the rod strikes a structural member of an aircraft the member will be severed. Such damage will usually cause flight failure unless the member is cut on an extremity such as a wing tip. After the rod reaches its maximum opening radius, the effectiveness of the rod falls off rapidly and can be ignored in kill probability estimates. (See figure 1B).

The lethal volume of a continuous rod warhead is then defined as that volume in which the target reference point must be located at burst if the target is to be lethally cut by the rod before the rod reaches its maximum opening radius R_L.[3]

C. PROBLEM STATEMENT

The major problem addressed in this thesis was to mathematically describe the set of locations of a target reference point relative to the warhead at the time of warhead detonation, such that the subsequent motion of the target and the rod would result in the rod cutting the target. This set was defined as the lethal volume of a continuous rod warhead.

[3]The contents of the section were paraphrased from M.C.Waddel, Surface-To-Air Guided Missile Systems, Applied Physics Laboratory TG 396, March 1961, pp. 37-38.



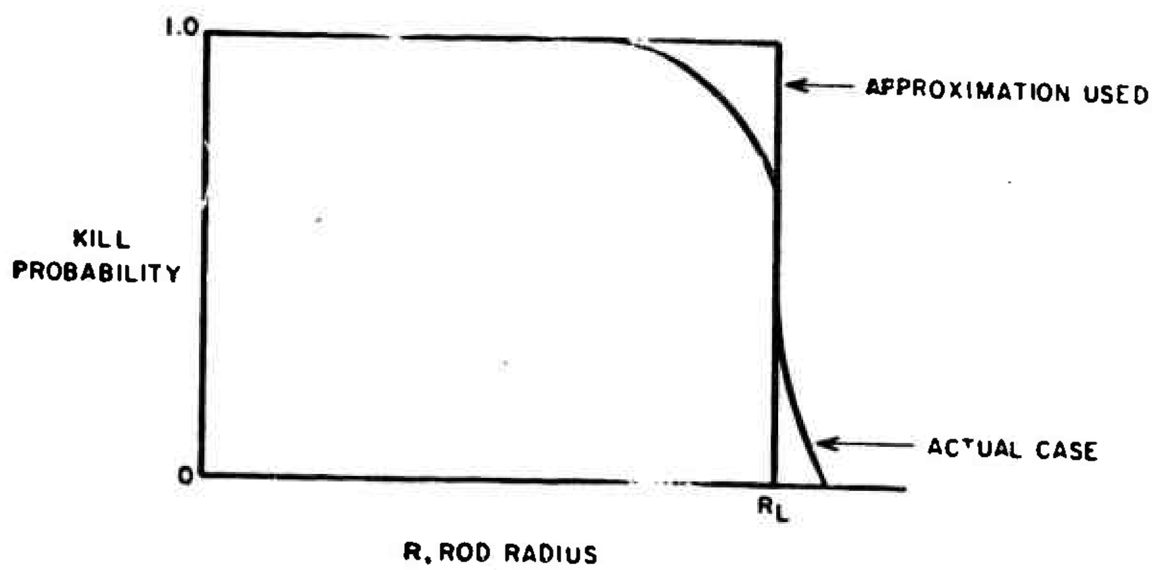


Figure 1A ROD EXPANSION RELATIVE TO THE GROUND

Since a necessary and sufficient condition for the target to be cut by the rod (target kill) is that the reference point be in the lethal volume at detonation, the probability of kill is the probability that the reference point is in the lethal volume at the time of warhead burst. A subsequent problem, then, was to obtain the probability content of the lethal volume and the variation in this probability content with model inputs.

D. IMPORTANCE OF STUDY

To date there has been very little effort devoted to an analytical description of the lethal volume of a continuous rod warhead. Nearly all effort in this area has been devoted to large scale simulations in which specific targets, missiles, and warheads are "flown" against each other under a given set of intercept or end game conditions. Using Monte Carlo techniques, a specific end game is run repeatedly until a probability of kill is obtained. The simulation technique has two major disadvantages. First, it is inherently expensive. Secondly, it fails to give the analyst an adequate "feel" for the relative importance of the variables under study. Because of the tremendous costs associated with the simulation approach, tradeoff studies among the inputs must be limited in scope.

On the other end of the spectrum of the tools that exist for the analysis of continuous rod warheads, are some elementary analytical models. Those models are accurate only for the simplest intercept geometries such as those in which the velocity vectors of both the target and the missile are along a straight line.[4]

[4] Ibid. pp. 51-53.

The major disadvantage of these models is their insensitivity to many of the parameters which are known to significantly affect the kill probability associated with a given end game.

The lethal volume model developed here was designed to bridge the gap between the simulation technique and simplified model; i.e., the lethal volume was intended to be sensitive to many of the variables which the simplified model is not and at the same time be substantially cheaper to use than the simulation technique. Because of the large decrease in costs, tradeoff studies performed with the lethal volume model can be much more complete than is currently possible with the simulation technique. For this reason, the lethal volume model is believed to be a significant new tool for the analysis of continuous rod warhead problems.

E. ORGANIZATION OF THESIS

Chapter II of the thesis contains the derivation of the lethal volume of a continuous rod warhead against a single, linear, non-rotating, and rigid line segment. The line segment is assumed to represent the fuselage or wing of an aircraft. The lethal volume is obtained for this target by examining the relative motion of the target and the rod with respect to the missile carrying the warhead. Chapter II is closed with examples of the shape of the lethal volume as a function of the relative location of the target and the missile carrying the continuous rod warhead at burst.

Chspter III contains expressions for the size and probability content of the lethal volume. A numerical scheme for the evalua-

tion of the expression of the probability content of the lethal volume is discussed and illustrated.

Chapter IV is devoted to an examination of the uses of the expressions for the size and probability content of the lethal volume derived in Chapter III. These uses are concerned with the variation in the size and probability content with the characteristics of the warhead, the missile carrying the warhead, and the target. A sample study concludes Chapter IV.

Chapter V is devoted to means of improving the warhead and target models used in the derivation. Particular attention is given to improving the target model so that is bears a closer resemblance to an aircraft.

CHAPTER II

DERIVATION

A. COORDINATE SYSTEMS AND DEFINITIONS

In this section two coordinate systems will be defined. One system will travel with the missile; therefore, the rod will remain in a plane in this coordinate system. Because of this fact the equations for the lethal volume are greatly simplified in this coordinate system. A second coordinate system will be defined such that it has the same orientation as the first but will be stationary. The second system will then be used to establish the motion between a rigid structural member (target) and the missile.

Let M be the point at the center of the warhead. A right-handed orthogonal coordinate system with its origin at M will now be established. First the x-axis is chosen such that it lies along the axis of the missile. The z-axis is then chosen such that it is perpendicular to the x-axis and such that the x-z axes lie in the vertical plane, with positive z "above" the origin. The y-axis is chosen to complete a right-handed system, (see Figure 2). Let $\bar{I}$, $\bar{J}$, and $\bar{K}$ be unit vectors directed along the positive x, y, and z axes respectively.

Since this coordinate system moves with the missile, the rod will move in the y-z plane. Furthermore, since the rod expands in a circle, the path of the rod is described by,

$$\bar{R} = (R \sin \theta)\bar{J} + (R \cos \theta)\bar{K}$$

where,

R is the radius of the circle,

θ is the angle between $\bar{R}$ and $\bar{K}$.

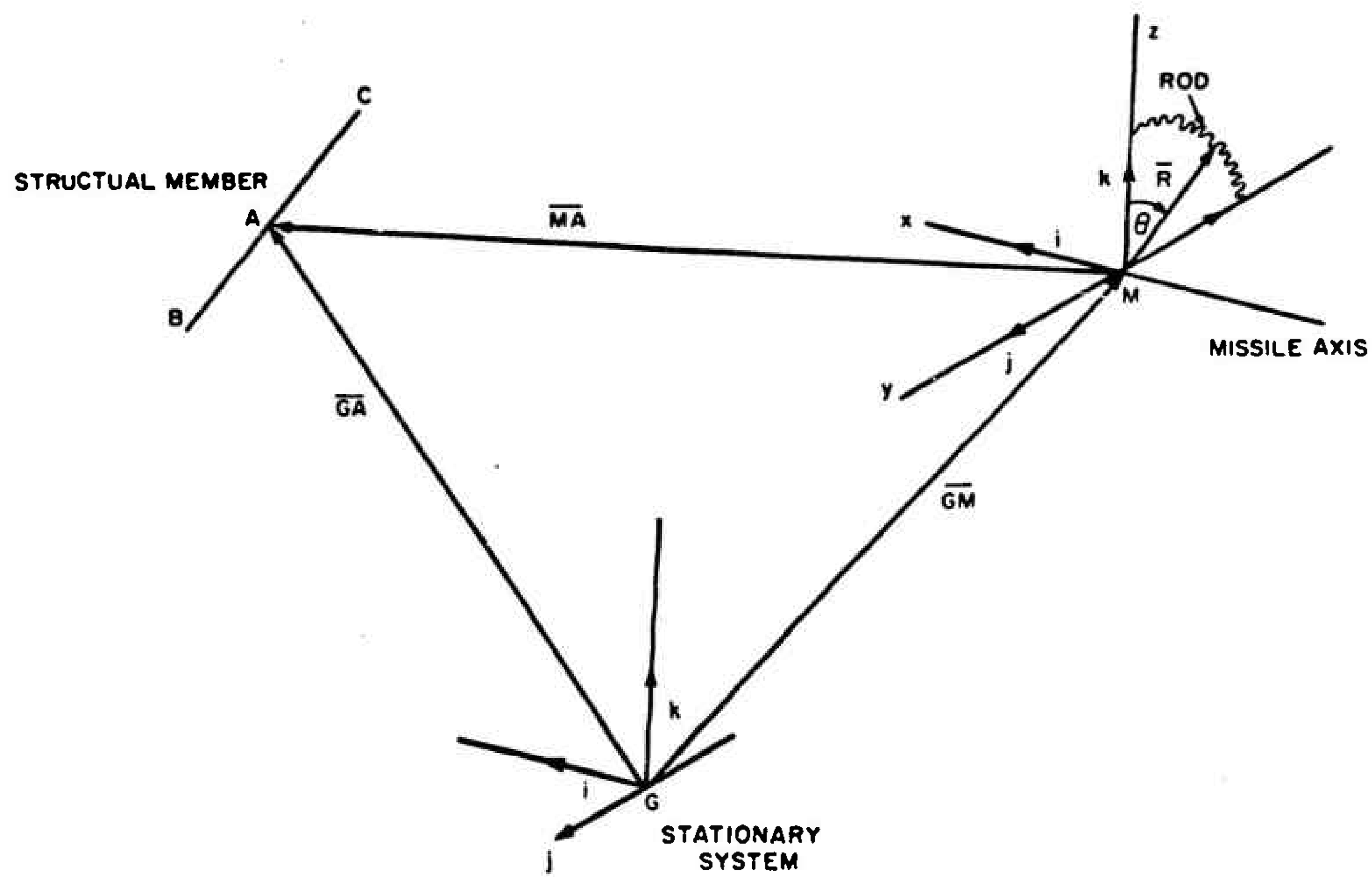


Figure 2 COORDINATE SYSTEMS

Note that $R \leq R_L$, the maximum opening radius.

Let G be the origin of a second coordinate system, having the same orientation as the above system but fixed relative to the ground. This coordinate system will be used only to establish the relative motion between M and the target.

Now consider a rigid linear structural member representing a major component of an aircraft such as the fuselage or a wing. Define B and C as the end points of the largest segment on the member such that a cut of the segment will be lethal to the member.

Define $\overline{BC}$ as the vector, having length equal to the distance from B to C and having the direction from B to C. Let the direction cosines of $\overline{BC}$ be ℓ_1, ℓ_2, and ℓ_3. $|\overline{BC}|$ will be referred to as the effective length of the member.

Let A be a point on the member between B and C. The velocity of A relative to M will now be established. Referring to figure 2, the vector $\overline{MA}$ is seen to be given by:

$$\overline{MA} = \overline{GA} - \overline{GM}.$$

Differentiating both sides with respect to time, t, gives

$$\frac{d\,\overline{MA}}{dt} = \frac{d\,\overline{GA}}{dt} - \frac{d\,\overline{GM}}{dt} \quad .$$

Defining $\frac{d\,\overline{MA}}{dt}$ as the velocity of A relative to M, denoted $\overline{V_{A/M}}$, gives

$$\overline{V_{A/M}} = \overline{V_{A/G}} - \overline{V_{M/G}} \quad .$$

Under the assumption that there is no rotation of the member during the period of interest, the velocity of all points of the member will be equal to $\overline{V_{A/M}}$. This assumption is adequate because of the extremely short time between warhead burst and the

time the rod reaches its maximum opening radius R_L. Denoting this common velocity by $\overline{V}_A$ gives

$$\overline{V_A} = \overline{V_{A/G}} - \overline{V_{M/G}} \quad . \tag{1}$$

Let the direction cosines of $\overline{V_{A/G}}$ be m_1, m_2, and m_3 and the absolute value of $\overline{V_{A/G}}$ be $|\overline{V_{A/G}}|$. In addition, let the direction cosines of $\overline{V_{M/G}}$ be n_1, n_2, and n_3 and let the absolute value of $\overline{V_{M/G}}$ be $|\overline{V_{M/G}}|$. Then

$$\overline{V_{M/G}} = |\overline{V_{M/G}}|\, n_1\, \overline{I} + |\overline{V_{M/G}}|\, n_2\, \overline{J} + |\overline{V_{M/G}}|\, n_3\, \overline{K}$$

and

$$\overline{V_{A/G}} = |\overline{V_{A/G}}|\, m_1\, \overline{I} + |\overline{V_{A/G}}|\, m_2\, \overline{J} + |\overline{V_{A/G}}|\, m_3\, \overline{K} \quad .$$

Substituting these values into equation (1)

$$\overline{V_A} = \left(|\overline{V_{A/G}}|\, m_1 - |\overline{V_{M/G}}|\, n_1\right)\overline{I} + \left(|\overline{V_{A/G}}|\, m_2 - |\overline{V_{M/G}}|\, n_2\right)\overline{J}$$
$$+ \left(|\overline{V_{A/G}}|\, m_3 - |\overline{V_{M/G}}|\, n_3\right)\overline{k}$$

or

$$V_A = V_x\, \overline{I} + V_y\, \overline{J} + V_z\, \overline{K}$$

where,

$$V_x = |\overline{V_{A/G}}|\, m_1 - |\overline{V_{M/G}}|\, n_1 \tag{2}$$

$$V_y = |\overline{V_{A/G}}|\, m_2 - |\overline{V_{M/G}}|\, n_2 \tag{3}$$

$$V_z = |\overline{V_{A/G}}|\, m_3 - |\overline{V_{M/G}}|\, n_3 \tag{4}$$

B. PARAMETRIC EQUATION OF THE LETHAL VOLUME

All positions, relative to M, at which the point B can be located at burst and the member still be cut by the rod between B and C will now be determined. The location of the point B will be defined such that any point of the member will be cut by the rod

at a radius R and angle θ . The selection of the point B as the target reference point is only for convenience since any target reference point would suffice.

The time for the rod to travel a distance R is given by

$$t = R/V_R$$

where,

V_R is the average velocity of rod to the distance R. If A is an arbitrary point on the member, the distance, with respect to M, that A travels during this time is $|\overline{d_A}|$ where

$$\overline{d_A} = t\,\overline{V_A} \quad .$$

Referring to figure 3, $\overline{R}$ is seen to be given by

$$\overline{R} = \overline{B}(A) + \overline{d_A} + \overline{BA}$$

where, $\overline{B}(A)$ is the set of coordinates of the reference point B (at the time of burst) such that the subsequent motion of the member and the rod will result in the rod being cut at the point A.

Since $\overline{BA}$ lies along $\overline{BC}$, $\overline{BA} = \lambda\overline{BC}$, where $\lambda = \frac{\overline{BA}}{\overline{BC}}$ and $0 \leq \lambda \leq 1$. Substituting and solving for $\overline{B}(A)$ gives

$$\overline{B}(A) = \overline{R} - \overline{d_A} - \lambda\overline{BC} \tag{5}$$

when $\lambda = 0$, .

$$\overline{B}(B) = \overline{R} - \overline{d_A}$$

and when $\lambda = 1$,

$$\overline{B}(C) = \overline{R} - \overline{d_A} - \overline{BC} \quad .$$

Therefore,

$$\overline{B}(C) = \overline{B}(B) - \overline{BC} \tag{6}$$

$$\overline{B}(A) = \overline{B}(B) - \lambda\overline{BC}$$

and

$$\overline{B}(A) = \overline{B}(B) - \lambda[\overline{B}(B) - \overline{B}(C)] \tag{7}$$

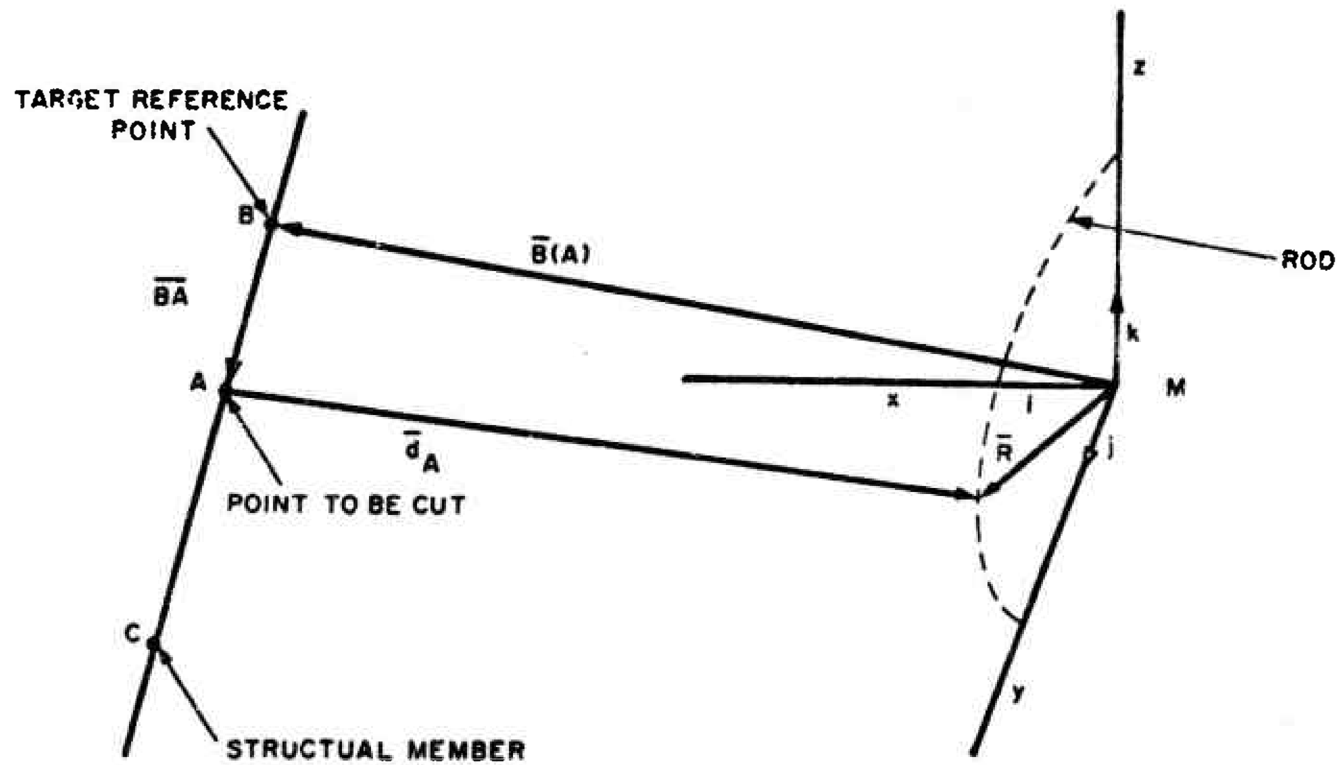


Figure 3. PATH OF TARGET TO BE CUT AT "A"

Two important observations concerning the volume generated by $\overline{B}(A)$ can be made from equations (6) and (7). Equation (7) shows that the volume generated by $\overline{B}(A)$ is composed of linear segments joining the surfaces generated by $\overline{B}(B)$ and $\overline{B}(C)$. Equation (6) shows that the surface generated by $\overline{B}(C)$ is identical to that generated by $\overline{B}(B)$ but is displaced by the vector $-\overline{BC}$. Hence, describing the lethal volume generated by $\overline{B}(A)$ is essentially reduced to describing the surface generated by $\overline{B}(B)$; i.e., the surface associated with cuts of the point B.

Substituting for $R, \overline{d_A}$, and $\overline{BC}$ in equation (7) gives,

$$\overline{B}(A) = \left[\left(\frac{R}{V_R}\right) V_x - \lambda|\overline{BC}|\,\ell_1\right]\overline{i} + \left[R \cos\theta + \left(\frac{R}{V_R}\right) V_y - \lambda|\overline{BC}|\,\ell_2\right]\overline{j}$$
$$+ \left[R \sin\theta + \left(\frac{R}{V_R}\right) V_z - \lambda|\overline{BC}|\,\ell_3\right]\overline{k} = \left[x_A, y_A, z_A\right] \quad . (8)$$

Equation (8) is the parametric equation of the lethal volume. When evaluated for all R, θ, and λ such that $0 \leq R \leq R_L$, $0 \leq \theta \leq 2\pi$, and $0 \leq \lambda \leq 1$ every point at which the reference B can be located at burst and the target be cut between B and C is generated. The volume generated by $\overline{B}(A)$ will be denoted V.

D. ELIMINATION OF PARAMETERS

Although equation (8) can be used to generate the lethal volume V, the elimination of the parameters R and θ will lead to a more efficient computation. Since no equation exists for reliably expressing V_R as a function of R, a completely closed form solution is impossible to obtain. However, a closed form solution can be obtained for any given R. In addition, under the assumption that V_R is constant, a closed form solution for the entire volume will

be obtained.

As pointed out in the preceding section, the problem of describing V is essentially equivalent to that of describing the surface generated by $\bar{B}(B)$. To find $\bar{B}(B)$, $\lambda = 0$ is substituted in equation (8) giving,

$$\bar{B}(B) = \left[x_B, y_B, z_B\right] = \left(\frac{R}{V_R}\right) V_x \,\bar{i} + \left[R \cos\theta + \left(\frac{R}{V_R}\right) V_y\right]\bar{j}$$

$$+ \left[R \sin\theta + \left(\frac{R}{V_R}\right) V_z\right]\bar{k}$$

or

$$x_B = \left(\frac{R}{V_R}\right) V_x \tag{9A}$$

$$y_B = R \cos\theta + \left(\frac{R}{V_R}\right) V_y \tag{9B}$$

$$z_B = R \sin\theta + \left(\frac{R}{V_R}\right) V_z \tag{9C}$$

Eliminating θ from equations (9B) and (9C) gives,

$$y_B^2 + z_B^2 - \frac{2V_z}{V_R} z_B R - \frac{2V_y}{V_R} y_B R = \frac{R^2}{V_R^2}\left(V_R^2 - V_y^2 - V_z^2\right) .$$

But from equation (9A) is obtained,

$$R = \frac{V_R}{V_x} x_B \quad , \quad \text{if } V_x \neq 0 \quad .$$

Substituting gives,

$$\frac{V_y^2 + V_z^2 - V_R^2}{V_x^2} x_B^2 + y_B^2$$

$$+ z_B^2 - \frac{2V_y}{V_x} x_B y_B - \frac{2V_z}{V_x} x_B z_B = 0 \tag{10A}$$

or in matrix form

$$B(B)\begin{bmatrix} H & -\frac{V_y}{V_x} & -\frac{V_z}{V_x} \\ -\frac{V_y}{V_x} & 1 & 0 \\ -\frac{V_z}{V_x} & 0 & 1 \end{bmatrix}\left[B(B)\right]^T \qquad (10B)$$

where,

$$H = (V_y^2 + V_z^2 - V_R^2)/V_x^2 \qquad (10C)$$

$$B(B) = [x_B, y_B, z_B]$$

$[B(B)]^T$ is the transpose of $B(B)$

If M denotes the matrix of coefficients in equation (10B), then

$$S(B) = [B(B)]\, M\, [B(B)]^T \qquad (11)$$

Where $S(B) = 0$ is the equation of the surface generated by $\bar{B}(B)$; i.e., the equation whose locus is the surface associated with cuts of the point B.

C. REDUCTION TO NORMAL FORM

In this section the normal form of S(B) will be obtained. First, the special case when both V_y and V_z are zero, will be treated. Then the canonical form for the case when V_z and V_y are not both zero will be obtained.

If an orthogonal matrix Q can be found such that,

$$QMQ^T = D \qquad (12)$$

where D is a diagonal matrix, then the substitution

$$B(B) = B'(B)\, Q \qquad (12A)$$

can be made in equation (11), giving,

$S(B) = B'(B)QM(B'(B)Q)^T = B'(B)QMQ^T[B'(B)]^T = B'(B)D[B'(B)]^T$

This will then be the required normal form of S(B) in terms of $\bar{B}'(B) = [x_B', y_B', z_B']$. The relationship between $\bar{B}(B)$ and $\bar{B}'(B)$ can be obtained from equation (12A).

First, consider the special case when both V_y and V_z are zero.

Then

$$M = \begin{bmatrix} -\frac{V_R^2}{V_z^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and,

$$S(B) = z_B^2 + y_B^2 - \frac{x_B^2(V_R^2)}{V_z^2} \quad ,$$

From equation (9C) the maximum value of z_B^2 is seen to be R^2. Dividing through by R^2 gives

$$\frac{z_B^2}{R^2} + \frac{y_B^2}{R^2} - \frac{x_B^2}{\left(\frac{RV_z}{V_R}\right)^2} = 0 \tag{13}$$

as normalized equation of the surface. For a given R, V_R is a constant and the surface associated with cuts of the point B is seen to be a right circular cone. If V_R is assumed to be constant for all R, then the cone for $R = R_L$ will contain the cones for all other values of R. Under this assumption, equation (13) takes the form,

$$\frac{z_B^2}{R_L^2} + \frac{y_B^2}{R_L^2} - \frac{x_B^2}{\left(\frac{R_L V_z}{V_R}\right)^2} = 0 \quad . \tag{14}$$

Next consider the case when neither V_y nor V_z is zero. Following the characteristic root - characteristic vector scheme for deriving Q, the characteristic roots were found to be,

$$\rho_1 = \frac{H + 1 + \sqrt{(H+1)^2 + 4\frac{V_R^2}{V_z^2}}}{2} > 0 \quad , \tag{15A}$$

$$\rho_2 = \frac{H + 1 - \sqrt{(H+1)^2 + 4\frac{V_R^2}{V_z^2}}}{2} < 0 \quad , \tag{15B}$$

and $$\rho_3 = 1 \tag{15C}$$

The characteristic vectors were found to be

$$\overline{q_1} = \left[1, \frac{1}{1-\rho_1}\frac{V_y}{V_z}, \frac{1}{1-\rho_1}\frac{V_z}{V_z}\right] \quad , \tag{15D}$$

$$\overline{q_2} = \left[1, \frac{1}{1-\rho_2}\frac{V_y}{V_z}, \frac{1}{1-\rho_2}\frac{V_z}{V_z}\right] \quad , \tag{15E}$$

and $$\overline{q_3} = \left[0, 1, \frac{-V_y}{V_z}\right] \quad . \tag{15F}$$

Therefore

$$Q = \begin{bmatrix} \overline{q_1}/|\overline{q_1}| \\ \overline{q_2}/|\overline{q_2}| \\ \overline{q_3}/|\overline{q_3}| \end{bmatrix}$$

Substituting Q in equation (12) gives

$$D = \begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence S(B) has form

$$S(B) = z_B'^2 + cx_B'^2 - d^2\, y_B'^2$$

where, $c = \sqrt{\rho_1}$ and $d = \sqrt{-\rho_2}$.

The tranaformation equations are given by

$$B'(B) = B(B)Q^T$$

or

$$x_B' = \left[x_B + \frac{V_y}{V_x}\left(\frac{1}{1-\rho_1}\right) y_B + \left(\frac{V_z}{V_x}\right)\left(\frac{1}{1-\rho_1}\right) z_B\right] / |\overline{q_1}| \tag{16A}$$

$$y_B' = \left[x_B + \frac{V_y}{V_x}\left(\frac{1}{1-\rho_2}\right) y_B + \left(\frac{V_z}{V_x}\right)\left(\frac{1}{1-\rho_2}\right) z_B\right] / |\overline{q_2}| \tag{16B}$$

$$z_B' = \left[y_B - \frac{V_y}{V_x} z_B\right] / |\overline{q_3}| \quad . \tag{16C}$$

Substituting from equations (9B) and (9C) into (16C) gives

$$z_B' = \left(R \cos\theta - \frac{V_y}{V_x} R \sin\theta\right)\sqrt{\frac{V_x^2 + V_y^2}{V_x^2}} \quad . \tag{17}$$

for a given R, the maximum value of z_B' is now desired in order to establish the general form of S(B). Differentiating with respect to θ and equating to zero gives,

$$\tan\theta = \frac{-V_y}{V_x}$$

Substituting into equation (17) yields

$$(z_B')_{Max} = \frac{R\,V_x^2}{(V_x^2 + V_y^2)} + \frac{R\,R\,V_y^2}{(V_x^2 + V_y^2)} \quad .$$

Therefore the normalized form of the surface associated with cuts of the point B is

$$\frac{z_B'^2}{R^2} + \frac{x_B'^2}{\left(\frac{R}{c}\right)^2} - \frac{y_B'^2}{\left(\frac{R}{d}\right)^2} = 0 \quad . \tag{18}$$

Note that when either V_y or V_x but not both is zero the vector $\overline{q_3}$ must be modified. When V_y is zero, then

$$z_B' = y_B \tag{19A}$$

and when $V_x = 0$,

$$\text{then} \quad z_B' = x_B \tag{19B}$$

In both cases $(z_B')_{Max}$ again equals R. However, equation (18) still holds since the formulas for c and d remain unchanged. If both V_y and V_z are zero the characteristic values change and equation (13) must be used. Again, the surface associated with cuts of the point B is seen to be a right cone or more specifically in this case a right elliptic cone. As before, if V_R is assumed to be constant, then the cone for $R = R_L$ will contain all other cones and equation (18) becomes

$$\frac{z_B'^2}{R_L^2} + \frac{x_B'^2}{\left(\frac{R_L}{c}\right)^2} - \frac{y_B'^2}{\left(\frac{R_L}{d}\right)^2} = 0$$

A definitive statement about the lethal volume V can now be made. Earlier the lethal volume was shown to be composed of all linear segments of length $|\overline{BC}|$ joining two identical surfaces generated by $\bar{B}(B)$ and $\bar{B}(C)$ where $|\overline{BC}|$ was the effective length of the member. These facts in conjunction with the foregoing statements about the surface generated by $\bar{B}(B)$, gives the following theorem:

> Under the assumption of a constant non-zero rod velocity $(V_R \neq 0)$ and a non-zero closing velocity projected on the missile axis $(V_x \neq 0)$, the lethal volume of a continuous rod warhead directed against a linear, rigid, and non-rotating structural member, is composed of linear segments connecting the surfaces of two right elliptic cones. Furthermore, these cones are identical in size and orientation. They are displaced from each other by a vector whose length is equal to the effective length of the member and whose direction is

given by the orientation of the member with respect to the axis of the missile carrying the warhead.

EXAMPLES

Before proceeding further with the derivation three simple examples will be given. The conditions associated with the examples will be referred to as end game conditions. End Game I is the simple case when the velocity vector of the target and the target velocity vector of the missile both lie along the same line. In End Game II the member is inclined at 45° to the axis of the missile. In End Game III the member is inclined at 90° to the axis of the missile. Table I defines the three end games in detail. The examples were treated parametrically except when complexity dictated otherwise.

TABLE I. Definition of End Games

Model Input[4]	End Game I	End Game II	End Game III
R_L	R_L	R_L	R_L
V_R	V_R (constant)	400 fps	400 fps
ℓ_1, ℓ_2, ℓ_3	ℓ_1, ℓ_2, ℓ_3	$\frac{-\sqrt{2}}{2}, 0, \frac{-\sqrt{2}}{2}$	0, 0, -1
$\lvert \overline{BC} \rvert$	$\lvert \overline{BC} \rvert$	$\lvert \overline{BC} \rvert$	$\lvert \overline{BC} \rvert$
m_1, m_2, m_3	-1, 0, 0	$\frac{-\sqrt{2}}{2}, 0, \frac{-\sqrt{2}}{2}$	0, 0, -1
n_1, n_2, n_3	1, 0, 0	1, 0, 0	1, 0, 0
$\lvert \overline{V_{A/G}} \rvert$	$\lvert \overline{V_{A/G}} \rvert$	300 fps	300 fps
$\lvert \overline{V_{M/G}} \rvert$	$\lvert \overline{V_{M/G}} \rvert$	200 fps	200 fps

[4]These terms are defined on pp.11-13

In End Game I both V_y and V_z are zero and equation (14) holds, consequently, the surface associated with cuts of the point B is a right circular cone of height $\dfrac{|V_z| R_L}{V_R}$

Figure 4 shows the lethal volume for this end game.

In End Game II the member is inclined at 45° to the axis of the missile. The velocity vector of the member and the velocity vector of the missile are aligned with the member and the axis of the missile, respectively. From equations (2), (3), and (4) are obtained,

$$V_x = -412 \text{ fps}, \; V_y = 0, \text{ and } V_z = -212 \text{ fps}.$$

From equations (10C), (15A), and (15B) are obtained,

$$H = -0.678, \; \rho_1 = 1.15 \text{ and } \rho_2 = -0.82 \quad .$$

Therefore S(B) has the form,

$$S(B) = \frac{z_B'^2}{R_L^2} + \frac{x_B'^2}{(0.94\, R_L)^2} - \frac{y_B'^2}{(1.1\, R_L)^2}$$

The translation equations (16A), (16B), and (19A) give,

$$x_B' = [x_B - 3.41\, z_B] \,/\, |\overline{q_1}| \quad ,$$

$$y_B' = [x_B + 0.283\, z_B] \,/\, |\overline{q_2}| \quad ,$$

$$z_B' = y_B \quad .$$

Using these axes the lethal volume was sketched in figure 5.

In End Game III the member is included at 90° to the axis of the missile. The velocity vectors at both the member and the missile are aligned with their respective axes. In this case,

$$V_x = 200 \text{ fps}, \; V_y = 0, \text{ and } V_z = -300 \text{ fps} \quad .$$

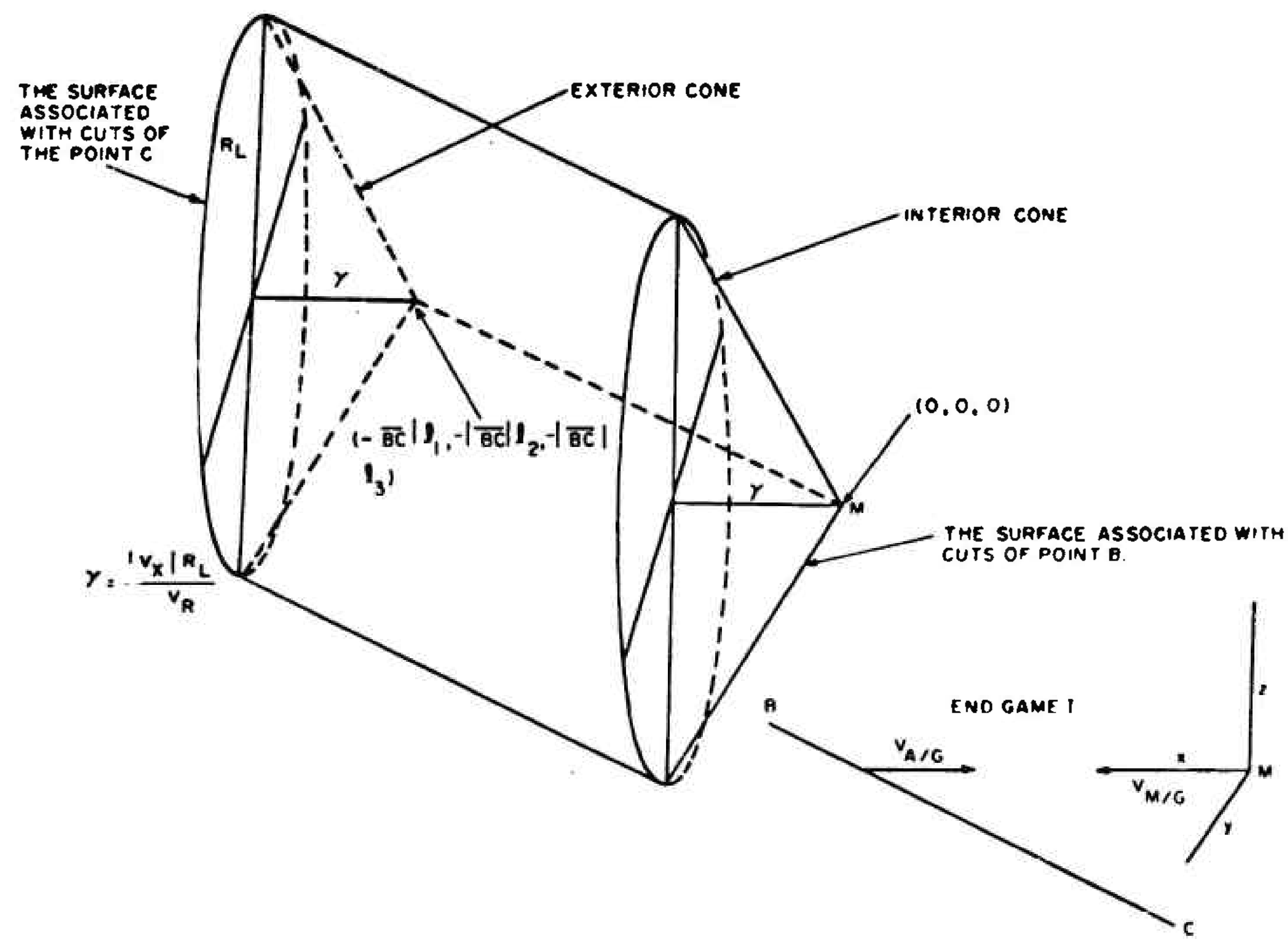


Figure 4. THE LETHAL VOLUME FOR END GAME I

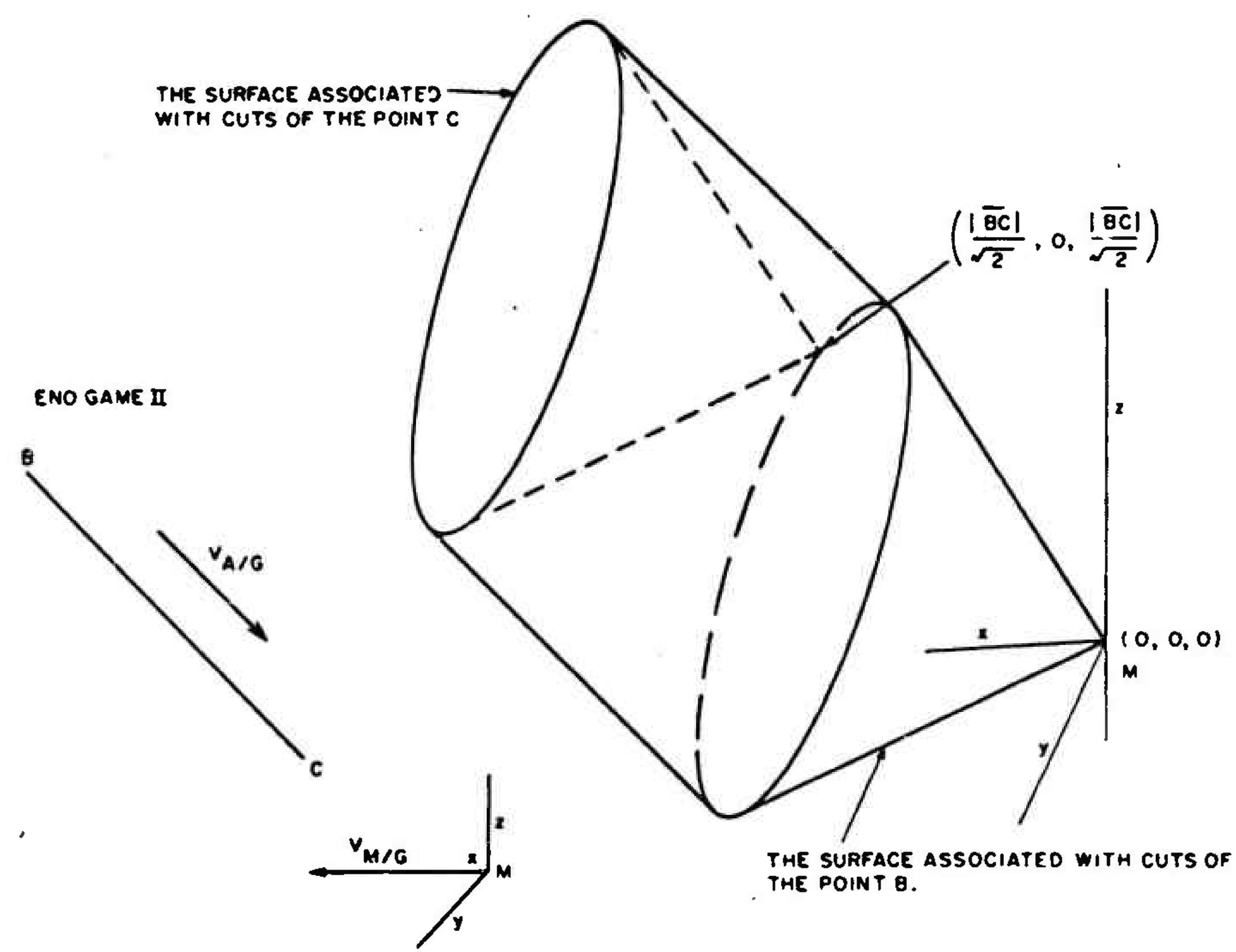


Figure 5 THE LETHAL VOLUME OF A STRUCTURAL MEMBER FOR END GAME II

In this case, $H = -1.75$, $\rho_1 = 1.66$, and $\rho_2 = -2.41$. Hence, S(B) has the form,

$$S(B) = \frac{z_B'^2}{R_L^2} + \frac{x_B'^2}{(.776\ R_L)^2} - \frac{y_B'^2}{(.644\ R_L)^2} = 0 \quad .$$

The translation equations for this end game are,

$$x_B' = [x_B - 2.28\ z_B] / |\overline{q_1}| \quad ,$$
$$y_B' = [x_B + 0.44\ z_B] / |\overline{q_2}|$$
$$\text{and } z_B' = y_B \quad .$$

Using these axes the lethal volume was sketched in figure 6.

Notice the marked difference in the characteristics of the lethal volume in End Game III as compared to either End Game I or End Game II. Figure 7 shows the transition in five stages. The figures show End Game I for various values of $\ell_1 = \cos\theta$.

First consider the cylinder formed by the lines joining the bases of the two cones in Case 1. As the angle θ increases, the apex of the cone contained in the cylinder (Case 2) approaches the side of the cylinder until the angle θ equals the semi-cone angle α (Case 3). As θ becomes greater than α the cone breaks through the side of the cylinder (Case 4). At $\theta = 90°$ the cone becomes completely outside the cylinder. (Case 5).

As θ becomes still larger the cone originally outside the cylinder starts to recede into the cylinder. At $\theta = 180°$ the volume is exactly as it appears in Case 1, except the cone outside the cylinder at $\theta = 0°$ is now inside the cylinder and vice-versa.

The condition for a cone to be completely contained in the cylinder when V_y and V_z are both zero, is

$$\alpha \geq \theta \quad .$$

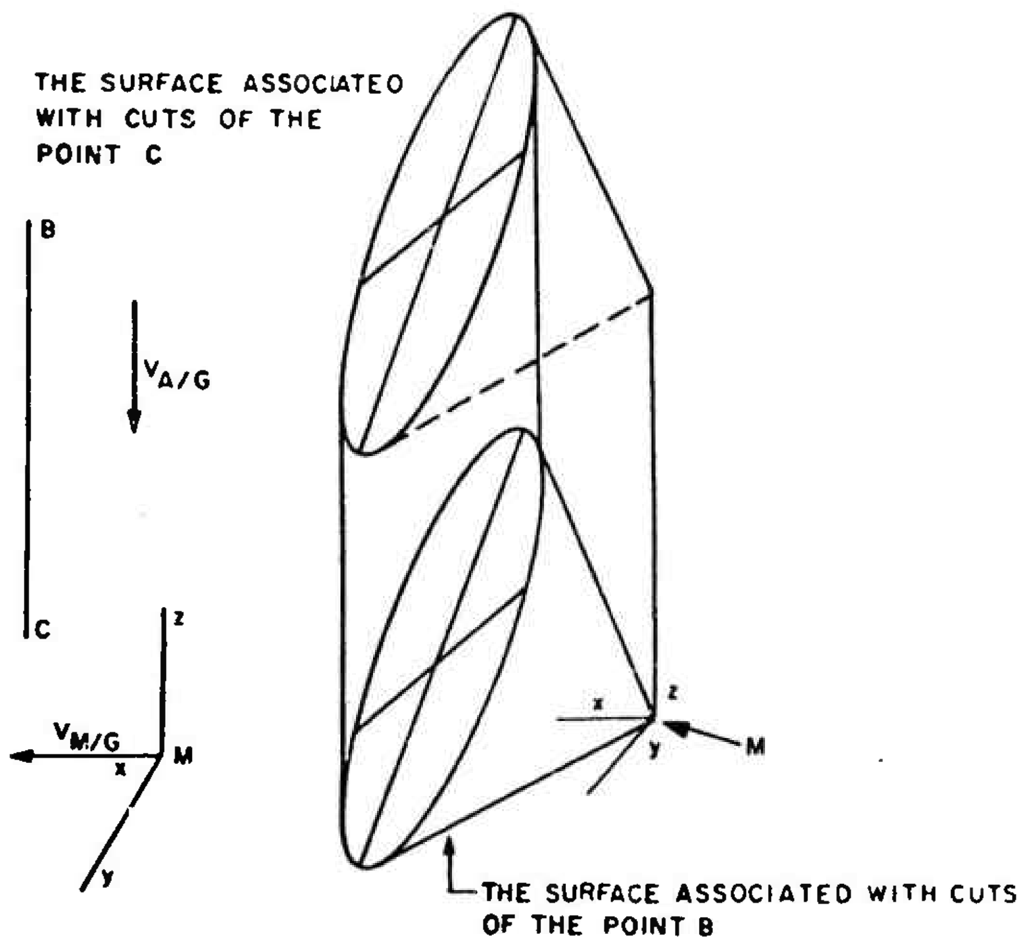


Figure 6 THE LETHAL VOLUME OF A STRUCTURAL MEMBER FOR END GAME III

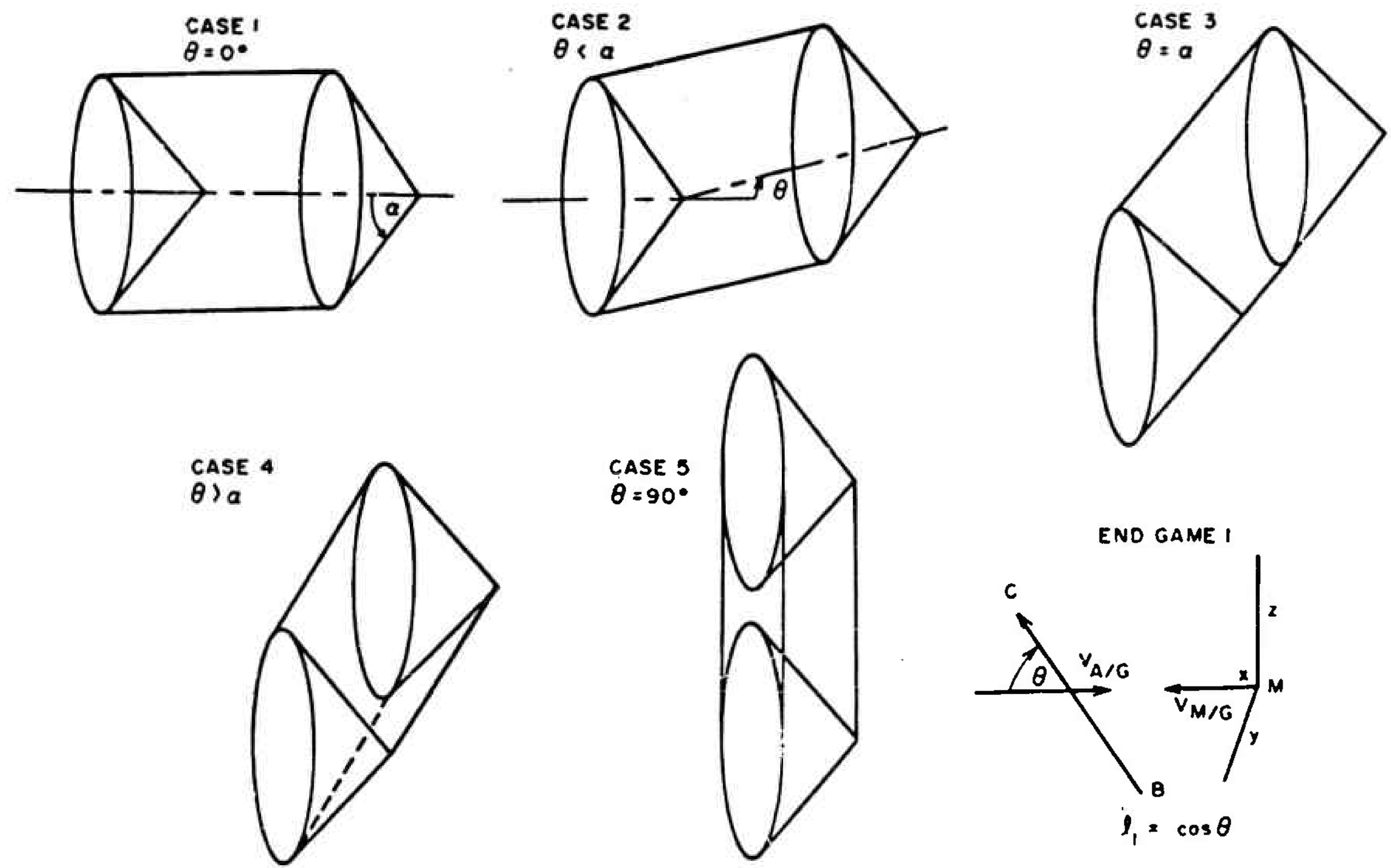


Figure 7 TRANSITION OF THE LETHAL VOLUME

Hence
$$\tan^{-1}\left(\frac{R}{\frac{|V_z|}{V_R}R}\right) \geq \cos^{-1}\left(\frac{|\overline{BC} \cdot i|}{|\overline{BC}|}\right) \tag{20A}$$

or
$$\frac{|\overline{BC} \cdot i|}{|\overline{BC}|} = \ell_1 \geq \frac{|V_z|}{\sqrt{V_R^2 + V_z^2}} \tag{20B}$$

is required condition.

Following a similar line of reasoning gives the condition when either V_y or V_z is not zero as

$$\left|\frac{\overline{BC} \cdot \overline{q_2}}{|\overline{q_2}|\,|\overline{BC}|}\right| \geq \frac{c}{\sqrt{c^2 + d^2}} \tag{20C}$$

CHAPTER III

MEASURES OF EFFECTIVENESS

A. INTRODUCTION

In this section expressions for the two measures of effectiveness discussed in the introduction, the size and probability content of the level volume, will be developed. The scope of this chapter will be limited by assuming a constant rod velocity and by considering the target to be a single linear member. In addition the results of this chapter are applicable only when equations (20B) or (20C) hold; i.e., only when one of the cones is completely contained in the cylinder. The latter restriction is not a serious limitation since the vast majority of end games satisfy these inequalities.

B. THE SIZE OF THE LETHAL VOLUME

From a computational standpoint, the size of the lethal volume is a very efficient measure of effectiveness. In this section, expressions will be given with which the size of the lethal volume can be evaluated when both V_y and V_z are zero and when either V_y or V_z is not zero.

Consider the lethal volume in figure 4. The lethal volume is composed of segments of length $|\overline{BC}|$ joining the surfaces of the two cones. The volume contained in one of the cones is seen to be interior to the lethal volume while the volume contained in the other cone is exterior to the lethal volume. Since these cones are identical in size, the volume contained in each is the same. The size of the lethal volume can, therefore, be gotten by finding the volume of the skewed elliptic cylinder obtained by joining the

bases of the two cones. A similar condition is seen to exist whenever one of the cones is completely contained in the cylinder; i.e., whenever equation (20B) or (20C) holds.

The volume of the skewed elliptic cylinder is given by,

$$V = \pi \, ab \, |\bar{L}| \, |\epsilon|$$

where,

a and b are the semimajor and semiminor axes of the ellipse, $\bar{L}$ is the vector connecting the center of the ellipses at the ends of the cylinder. ϵ is the cosine of the angle between $\bar{L}$ and the vector perpendicular to the surface of the ellipse. This fact will now be related to the size of the lethal volume.

Let V denote the lethal volume and $|V|$ denote the size of this lethal volume. When $V_y = V_z = 0$, $\overline{BC}$ corresponds to the vector $\bar{L}$, ℓ_1 corresponds to ϵ, and $a = b = R_L$. Hence,

$$|V| = \pi \, |\overline{BC}| \, |\ell_1 R_L^2| \quad , \tag{21}$$

subject to,

$$\ell_1 \geq \frac{|V_x|}{\sqrt{V_R^2 + V_x^2}}$$

When V_y and V_z are not both zero, the cones revolve about the y' axis. The vector $\overline{q_3}$, (15E), is, therefore, perpendicular to the surface of the ellipse. In this case, $\overline{BC} = \bar{L}$, $a = R_L$, and $b = R_L/c$; hence,

$$|V| = \frac{\pi R_L^2}{c}\left(\frac{|\overline{BC} \cdot \overline{q_3}|}{|\overline{BC}|\,|\overline{q_3}|}\right) = \frac{\pi R_L^2}{c} \; \frac{|\overline{BC} \cdot \overline{q_3}|}{|\overline{q_3}|} \tag{22}$$

Subject to,

$$\frac{|\overline{BC} \cdot \overline{q_3}|}{|\overline{q_3}|\,|\overline{BC}|} \geq \frac{c}{\sqrt{c^2 + d^2}} \quad . \tag{20C}$$

As an example, consider one of the lethal volumes of the preceding chapter. For End Game II, equation (20C) holds, and,

$$|V| = \frac{\pi R_L^2 \, |\overline{BC}|}{\sqrt{1.12}} (0.872) = 0.825 \, \pi R_L^2 \, |\overline{BC}| \quad ,$$

since,

$$\frac{|\overline{q_4} \cdot \overline{BC}|}{|\overline{q_3}|} = |\overline{BC}| \left[\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right] \left[.965, 0, .266\right] = 0.872 \, |\overline{BC}| \ .$$

An important property of the size of the lethal volume can now be stated:

> For all end games where the inequality (20B) or the inequality (20C) holds, the size of the lethal volume for cuts of a single linear structural member is a maximum when the velocity vectors of the target and the missile are parallel, ($V_y = V_z = 0$), and the axis of both the target and the missile lie along these vectors ($|\ell_1| = 1$).

This follows directly from equations (21) and (22) since and the maximum value of $|\ell_1| = 1$.

THE PROBABILITY CONTENT OF THE LETHAL VOLUME

1. Introduction

An approach similar to that used to calculate the volume $|V|$ can also be used to find the probability content of V. That is, the probability content of V can be considered to be equal to the probability content of an elliptic cylinder plus the probability content of an elliptic cone less the probability content of an identical cone. Unfortunately, the probability content of the two cones are not usually equal; hence, they do not cancel as they did

in the calculation of V . In general, the probability content of V, P_k , is given by,

$$P_k = P_1 + P_2 - P_3$$

where,

P_1 is the probability content of the elliptic cylinder,

P_2 is the probability content of the cone interior to the lethal volume,

and P_3 is the probability content of the cone exterior to the lethal volume.

2. Background

Figure 8 shows a skewed elliptic cylinder and a right elliptic cone together with a coordinate system (t', u', v'). The coordinate system is chosen such that the t' axis lies along the axis of the cone. The u' and v' axes are chosen to be parallel to the major and minor axes, respectively, of the elliptic base of the cone. In addition, the base of the cylinder is assumed to be parallel to the u'-v' plane. The base of the cylinder nearest the origin is assumed to be centered at the point t_o' and the apex of the cone nearest the origin is assumed to be located at t_1' . Also shown is the mean $(\bar{t}', \bar{u}', \bar{v}')$ of an arbitrary frequency function $f(t', u', v')$.

Expressions for the probability content of these two geometrical figures under the frequency function $f(t', u', v')$ will now be developed. In general the probability content of a volume under the frequency function $f(t', u', v')$ is given by,

$$P = \iiint_{Vol} f(t', u', v')\, dt'du'dv' \quad .$$

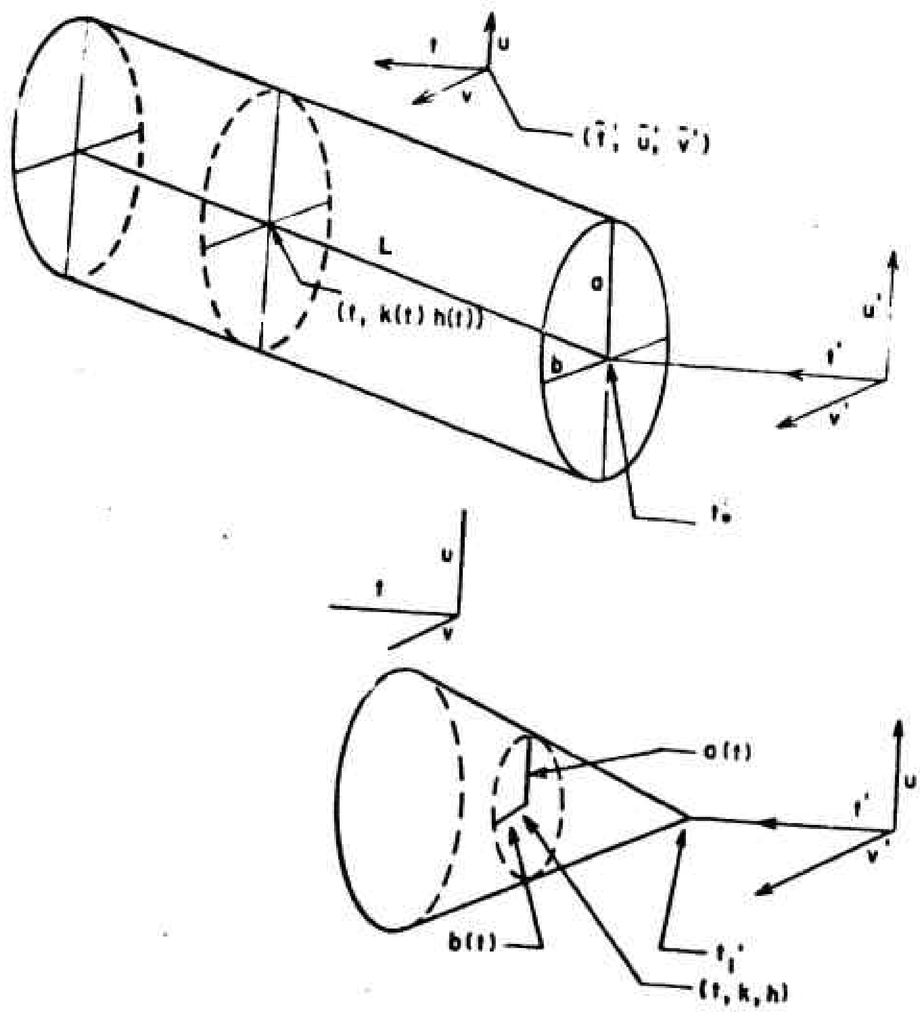


Figure 8. COORDINATES FOR DETERMINING KILL PROBABILITY

The limits of integration are easier to visualize if the origin is transformed to the mean $(\bar{t}', \bar{u}', \bar{v}')$, so let

$$t = t' - \bar{t}' \quad ,$$
$$u = u' - \bar{u}' \quad ,$$
$$\text{and } v = v' - \bar{v}' \quad .$$

The expression for P then becomes

$$P = \iiint\limits_{Vol} f(t, u, v)\, dtdudv \quad ,$$

since the Jacobian of the transformation is one.

First consider the cylinder. The axis of the cylinder is formed by the vector

$$\bar{L} = |\bar{L}|[\epsilon_1, \epsilon_2, \epsilon_3]$$

where, ϵ_1, ϵ_2, and ϵ_3 are the direction cosines of $\bar{L}$ with respect to the (t', u', v') coordinate system. The semi-minor and semi-major axes of the elliptic cross sections of the cylinder are denoted by a and b respectively. The equation of this cross section is given by,

$$\frac{[u-k(t)]^2}{a^2} + \frac{[v-h(t)]^2}{b^2} - 1 = 0 \qquad (23)$$

where,

k(t) and h(t) are the u and v coordinates, respectively, of the center of the cross section taken at the point t.

By similar triangles it is seen that

$$k(o) = \frac{\epsilon_2}{\epsilon_1}(t' - t_o') - \bar{u}'$$

and that,

$$k(t) = \frac{\epsilon_2}{\epsilon_1} t + k(o) = \frac{\epsilon_2}{\epsilon_1}(t + \bar{t}' - t_o') - \bar{u}' \quad .$$

Analogously,

$$h(t) = \frac{\epsilon_2}{\epsilon_1}(t + \bar{t}' - t_o) - \bar{v}'$$

Solving equation (23) for v gives,

$$v(t, u) = h(t) \pm \frac{b}{a}\sqrt{a^2 - [u-k(t)]^2} \quad .$$

The probability content of the cylinder then can be expressed as,

$$P = \int_{L_{11}}^{U_{11}} \int_{L_{12}}^{U_{12}} \int_{L_{13}}^{U_{13}} f(t, u, v)\, dtdudv \qquad (24)$$

where,

$$U_{11} = |\Sigma| \epsilon_1 + t_o' - \bar{t}' \quad ,$$

$$L_{11} = t_o' - \bar{t}' \quad ,$$

$$U_{12} = k(t) + a \quad ,$$

$$L_{12} = k(t) - a \quad ,$$

$$U_{13} = h(t) + \frac{b}{a} \sqrt{a^2 - [u-k(t)]^2} \quad ,$$

and $$L_{13} = h(t) - \frac{b}{a} \sqrt{a^2 - [u-k(t)]^2} \quad .$$

A similar approach can be used in deriving the probability content of the cone. In this case,

$$k(t) = k(t_o - t_1) = -\bar{u}'$$

$$h(t) = h(t_o - t_1) = -\bar{v}'$$

and are, hence, constants. However, a(t) and b(t), the semi-major and semi-minor axes of the elliptic cross sections of the cones, vary with t. If the height of the cone is designated by γ , then

$$\frac{\gamma}{a} = \frac{|\, t + (t' - t_1)\,|}{a(t)}$$

$$\text{or } a(t) = \frac{a}{\gamma} |\ (t + t' - t_1)\ | \ .$$

Analogously,

$$b(t) = \frac{b}{\gamma} |\ (t' - t_1')\ | \ .$$

For the cone, then

$$v(t, u) = -v' \pm \frac{b(t)}{a(t)} \sqrt{a^2(t) - [u+u']^2}$$

The probability content of the cone, P_2, then can be expressed as

$$P_2 = \int_{L_{21}}^{U_{21}} \int_{L_{22}}^{U_{22}} \int_{L_{23}}^{U_{23}} f(t, u, v) \, dtdudv \tag{25}$$

where,

$$U_{21} = \gamma + t_1' + \bar{t}' \quad ,$$

$$L_{21} = t' - \bar{t}' \quad ,$$

$$U_{22} = -\bar{u}' + a(t) \quad ,$$

$$L_{22} = -\bar{u}' - a(t) \quad ,$$

$$U_{23} = -\bar{v}' + \frac{b(t)}{a(t)} \sqrt{a^2(t) - [u + \bar{u}']^2} \quad ,$$

and $$L_{23} = -\bar{v}' - \frac{b(t)}{a(t)} \sqrt{a^2(t) - [u + u']^2} \quad .$$

3. An Expression for the Probability Content of the Lethal Volume

The results of the latter derivations, together with the results of Chapter II, will now be used to derive an expression for the probability content of the lethal volume.

In Chapter II, the feasibility of generating an orthogonal coordinate system x_B', y_B', z_B' with the following properties was demonstrated:

(a) The x_B' axis was shown to be parallel to the minor axis of the cones composing the lethal volume.

(b) The y_B' axis was shown to be parallel to the axis of this cone and is perpendicular to the base of the cone.

(c) The z_B' axis was shown to be parallel to the major axis of this ellipse.

A lethal volume is shown in figure 9, together with such a coordinate system constructed with its origin at the apex of the interior cone. This coordinate system meets all the requirements specified for the (t', u', v') coordinate system.

Recall that the height of the cone was shown to be $\frac{R_L}{d}$, that the major axis of the base of the cone was shown to be R_L, and that the minor axis of the base of the cone was shown to be $\frac{R_L}{c}$.[1]

Furthermore, if the vector $\overline{BC}$ is restricted to be directed in the positive y_B' direction then the vector $\overline{BC}$ corresponds to the vector $\overline{L}$. This restriction does not limit the applicability of the derivation in any way since the choice of points B and C and hence direction of the vector $\overline{BC}$ is completely arbitrary.

In Chapter III, $\overline{BC} = |\overline{BC}|\,(\ell_1, \ell_2, \ell_3)$, was given relative to the (x, y, z) coordinate system. To be applicable here, $\overline{BC}$ must be transformed to the (x_B', y_B', z_B') system by equations (16A), (16B), and (16C). The transformed and restricted vector $\overline{BC}$ will be denoted by $\overline{BC}' = |\overline{BC}|\,(\ell_1', \ell_2', \ell_3')$. Then $\epsilon_1, \epsilon_2, \epsilon_3$ corresponds to ℓ_2', ℓ_3', and ℓ_1' respectively.

Also shown in figure 9, is the location of the mean $(\overline{x}_B', \overline{y}_B', \overline{z}_B')$ of the frequency function $f(x_B', y_B', z_B')$. As before the origin is translated to the mean by letting,

$$t = y_B' - \overline{y}_B' \quad ,$$

$$u = z_B' - \overline{z}_B' \quad ,$$

$$\text{and} \quad v = x_B' - \overline{x}_B' \quad ,$$

[1] See equation (17), p. 20.

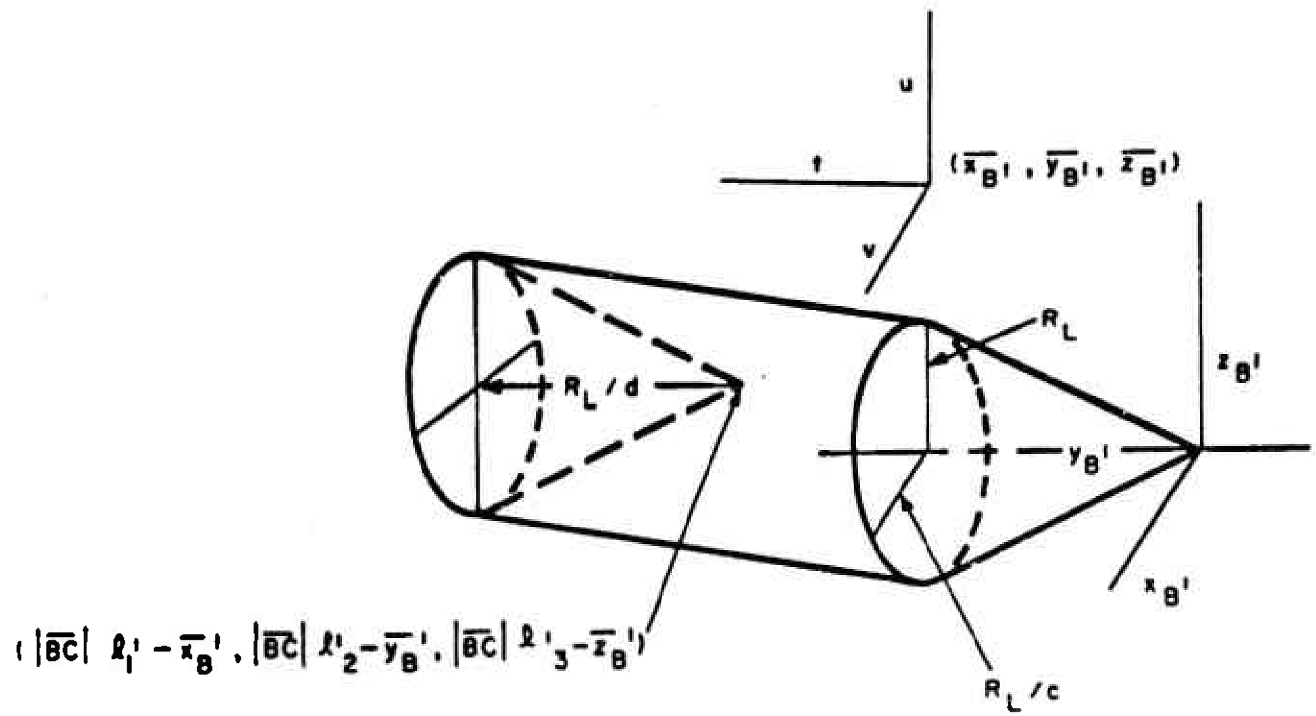


Figure 9. COORDINATES FOR DETERMINING THE PROBABILITY CONTENT OF THE LETHAL VOLUME

The expressions for the probability content, P_1, of the cylinder joining the bases of the two cones can now be easily obtained. Note that t_o, the nearest point to the origin of the cylinder is given by the height of the cone $\frac{R_L}{d}$. Using equation (22), P_1, is given by,

$$P_1 = \int_{L_{11}}^{U_{11}} \int_{L_{12}}^{U_{12}} \int_{L_{13}}^{U_{13}} f(t, u, v)\, dtdudv$$

where,

$$U_{11} = |\overline{BC}|\, \ell_2' + \frac{R_L}{d} - \overline{y_B}' ,$$

$$L_{11} = \frac{R_L}{d} - \overline{y_B}' ,$$

$$U_{12} = R_L + k(t) ,$$

$$L_{12} = R_L - k(t) ,$$

$$U_{13} = h(t) + \frac{1}{c}\sqrt{R_L^2 - [u-k(t)]^2} ,$$

$$L_{13} = h(t) - \frac{1}{c}\sqrt{R_L^2 - [u-k(t)]^2} ,$$

$$k(t) = \frac{\ell_3'}{\ell_2'}\left(t + \overline{y_B}' - \frac{R_L}{d}\right) - \overline{z_B}' , \tag{26A}$$

and

$$h(t) = \frac{\ell_3'}{\ell_2'}\left(t + \overline{y_B}' - \frac{R_L}{d}\right) - \overline{x_B}' . \tag{26B}$$

The probability content of the cones can be found in a similar fashion. For the interior cone, the distance from the origin of the (x_B', y_B', z_B') system to the apex of the cone, t_1', is zero. The probability content of the interior cone, P_2, is given by,

$$P_2 = \int_{L_{21}}^{U_{21}} \int_{L_{22}}^{U_{22}} \int_{L_{23}}^{U_{23}} f(t, u, v)\, dtdudv$$

where,

$$U_{21} = \frac{R_L}{d} - \overline{y_B}' \quad ,$$

$$L_{21} = -\overline{y_B}' \quad ,$$

$$U_{22} = \overline{z_B}' + a(t) \quad ,$$

$$L_{22} = \overline{z_B}' - a(t) \quad ,$$

$$U_{23} = -\overline{x_B}' + \frac{b(t)}{a(t)} \sqrt{a^2(t) - [u+\overline{z_B}']^2} \quad ,$$

$$L_{23} = -\overline{x_B}' - \frac{b(t)}{a(t)} \sqrt{a^2(t) - [u+\overline{z_B}']^2} \quad ,$$

$$a(t) = d \,|\, \overline{y_B}' + t \,| \quad ,$$

and $$b(t) = \frac{d}{c} \,|\, \overline{y_B}' + t \,| \quad .$$

The form developed for the probability content of the interior cone is sufficiently general to be used to express the probsbility content of the exterior cone. For the exterior cone, the distsnce from the origin of the (x_B', y_B', z_B') system to the apex of the cone is given by,

$$t_1 = |\overline{BC}|\, \ell_2$$

The quantities k(t) and h(t) are again constsnts with,

$$k(t) = k(\,|\overline{BC}|\, \ell_2' - \overline{y_B}') = |\overline{BC}|\, \ell_3' - \overline{z_B}' \quad ,$$

and $$h(t) = h(\,|\overline{BC}|\, \ell_2' - \overline{x_B}') = |\overline{BC}|\, \ell_1' - \overline{x_B}' \quad .$$

The probability content of the exterior cone, P_3, is given by,

$$P_3 = \int_{L_{31}}^{U_{31}} \int_{L_{32}}^{U_{32}} \int_{L_{33}}^{U_{33}} f(t, u, v) \;\; dtdudv$$

$$U_{31} = \left(\frac{R_L}{d}\right) + |\overline{BC}|\, \ell_2' - \overline{y_B}' \quad ,$$

$$L_{31} = |\overline{BC}|\, \ell_2' - \overline{y_B}' \quad ,$$

$$U_{32} = |\overline{BC}|\, \ell_3' - \overline{z_B}' + a(t) \quad ,$$

$$L_{32} = |\overline{BC}|\, \ell_2' - \overline{z_B}' - a(t) \quad ,$$

$$U_{33} = |\overline{BC}|\, \ell_1' - \overline{x_B}' + d\sqrt{a^2(t) - [u+\overline{z_B}'\, |\overline{BC}|\, \ell_3']^2} \quad ,$$

$$L_{33} = |\overline{BC}|\, \ell_1' - \overline{x_B}' - d\sqrt{a^2(t) - [u+\overline{z_B}'\, |\overline{BC}|\, \ell_3']^2} \quad ,$$

$$a(t) = d\,|\,t - \left(\frac{R_L}{d} + |\overline{BC}|\, \ell_2' - \overline{y_B}'\right)| \quad ,$$

and $$b(t) = \frac{d}{c}\,|\,t - \left(\frac{R_L}{d} + |\overline{BC}|\, \ell_2' - \overline{y_B}'\right)| \quad .$$

After evaluating the integrals for P_1, P_2, and P_3, the probability content of lethal volume P_k, is given by,

$$P_k = P_1 + P_2 - P_3 \quad .$$

The probability content formulas derived thus far are applicable only when V_y or V_z is not zero and when, in addition, the inequality (20C) holds.

The equations for the probability content of the lethal volume when both V_y and V_z are zero follow in a analogous fashion. In this case, the coordinate system (x, y, z) is equivalent to the (t', u', v') system. If the mean of the frequency function f(x, y, z) is $(\bar{x}, \bar{y}, \bar{z})$, the substitution,

$$t = x - \bar{x} \quad ,$$

$$u = y - \bar{y} \quad ,$$

and $$v = z - \bar{z} \quad .$$

will translate the origin to the mean.

Recall that in this case[2] the cones are right circular cones with

[2]See equation (13), p. 18

heights $\left|\frac{V_x}{V_R}\right| R_L$. Since the cones are circular, a(t) equals b(t) and in the cylinder $a = b = R_L$.

Again the vector $\overline{BC}$ must be restricted before it will correspond to the vector $\bar{L}$. If BC is oriented in the direction of positive x then $\overline{BC} = |\overline{BC}|\ (\ell_1, \ell_2, \ell_3)$ corresponds to the vector $\bar{L}$. $\overline{BC}$ is relative to the (x, y, z) coordinate system and ℓ_1, ℓ_2 and ℓ_3 corresponds to ϵ_1, ϵ_2 and ϵ_3 respectively. Equations (24) and (25) then can be used for the probability content of the lethal volume in this case with,

$$\bar{L} = \overline{BC}$$

$$\epsilon_1, \epsilon_2, \epsilon_3 = \ell_1, \ell_2, \ell_3 \text{ respectively}$$

$$t', \bar{u}', \bar{v}' = \bar{x}, \bar{y}, \bar{z} \text{ respectively}$$

$$t_o = \gamma = \frac{|V_x|}{V_R} R_L$$

$$a = b = R_L$$

$$t_1{}' = 0 \text{ (interior cone)}$$

$$t_1{}' = |\overline{BC}|\ \ell_1 \text{ (exterior cone)}$$

4. Evaluation of the Integrals

A scheme for both hand evaluation and computer evaluation of the integrals P_1, P_2, and P_3 will now be discussed. The scope of this section will be limited to the evaluation of the integrals with only the trivariate normal frequency function,

$$f(t,u,v) = \left(\frac{1}{2\pi}\right)^{3/2} \frac{1}{\sigma_t \sigma_u \sigma_v} e^{-\left(\frac{t^2}{\sigma_t^2} + \frac{t^2}{\sigma_u^2} + \frac{t^2}{\sigma_v^2}\right)}$$

The integral for the probability content of the cylinder or the cones over the trivariate normal distribution f(t, u, v),

given above, can be expressed as,

$$P = \int_{L_1}^{U_1}\int_{L_2}^{U_2}\int_{L_3}^{U_3} f(t, u, v)\, dtdudv \tag{27}$$

where,

$$U_1 \text{ and } L_1 \text{ are constants} \quad ,$$

$$U_2 = k(t) + a(t) \quad ,$$

$$L_2 = k(t) - a(t) \quad ,$$

$$U_3 = h(t) + \frac{b(t)}{a(t)}\sqrt{a^2(t) - [u-k(t)]^2} \quad ,$$

and

$$L_3 = h(t) - \frac{b(t)}{a(t)}\sqrt{a^2(t) - [u-k(t)]^2} \quad .$$

To evaluate this integral, first make the normalizing transformation,

$$t' = t/\sigma_t$$

$$u' = u/\sigma_u$$

$$v' = v/\sigma_v$$

Equation (27) becomes,

$$P = \left(\frac{1}{2\pi}\right)^{3/2} \int_{L_1'}^{U_1'}\int_{L_2'}^{U_2'}\int_{L_3'}^{U_3'} f(t', u', v')\, dtdudv$$

where,

$$U_1' = U_1/\sigma_t \quad ,$$

$$L_1' = L_1/\sigma_t \quad ,$$

$$U_2' = k'(t') + a'(t') \quad ,$$

$$L_2' = k'(t') - a'(t') \quad ,$$

$$U_3' = h'(t') + \frac{b'(t')}{a'(t')}\sqrt{[a'(t')]^2 - [u'-k'(t')]^2} \quad ,$$

$$L_3' = h'(t') - \frac{b'(t')}{a'(t')}\sqrt{[a'(t')]^2 - [u'-k'(t')]^2} \quad ,$$

$$k' = k/\sigma_u \quad ,$$

$$h' = h/\sigma_v \quad ,$$

$$a' = a/\sigma_u \quad ,$$

and

$$b' = b'/\sigma_v \quad .$$

Since this transformation maps ellipses into ellipses and straight lines into straight lines, the problem is still to evaluate the integral P over a right elliptic cone and skewed elliptic cylinder.

A computer program suitable for the evaluation of P is readily available for the IBM 7090 computer.[3] This routine computes the integral, P, using a Hasting's approximation[4] for the inner integral then sums by a step by step process.

The integral P, can be evaluated in terms of tabulated functions only if $k'(t'), h'(t')$, are constants and $a'(t)$ and $b'(t)$ are equal. Since $a'(t')$ and $b'(t')$ are always different for the cones when V_y or V_z is not zero, numerical integration must usually be used to evaluate the probability content of the cones. With the cylinder, $a(t)$ and $b(t)$ are always constants and when either V_y or V_z is not zero, equations (26A) and (26B) show that $k'(t')$ and $h'(t')$ are also constants if $\ell_3' = \ell_1' = 0$. The corresponding condition for the case when both V_y and V_z are zero is $\ell_2 = \ell_3 = 0$.

When $a'(t'), b'(t'), k'(t')$ and $h'(t')$ are constants then

[3]Gittleman, Ruth, Floating Point (n) Variate Probability Integral, IBM SHARE ROUTINE 1384, November 2, 1962.

[4]Hastings, Cecil, Jr., Approximation for Digital Computers, Princeton University Press, 1955, p. 167.

the integral P can be written as

$$P = \left(\frac{1}{2\pi}\right)^{1/2} \int_{L_1'}^{U_1'} e^{-\frac{1}{2}t'^2}\, dt' \int_{k'-a'}^{k'+a'} \int_{h' - \frac{b'}{a'}\sqrt{a'^2 - (u'-k')^2}}^{h' + \frac{b'}{a'}\sqrt{a'^2 - (u'-k')^2}} \left(\frac{1}{2\pi}\right) e^{-\frac{1}{2}(u'+v')^2} du'\, dv'$$

or

$$P = \Delta\Omega\; E(a', b', k', h') \tag{28A}$$

where,

$$\Delta\Omega = \Omega(U_1') - \Omega(L_1') \quad , \tag{28B}$$

$$\Omega(X_o) = \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{X_o} e^{-\frac{1}{2}t'^2} dt' \tag{28C}$$

and

$$E(a', b', k', h') = \frac{1}{2\pi} \int_{k'-a'}^{k'+a'} \int_{h' - \frac{b'}{a'}\sqrt{a'^2 - (u'-k')^2}}^{h' + \frac{b'}{a'}\sqrt{a'^2 - (u'-k')^2}} e^{-\frac{1}{2}(u'^2+v'^2)}\, du'\, dv'$$

$\Omega(X)$ is the integral of normal distribution function and is tabulated in numerous books.

$E(a', b', k', h')$ is the integral of the bivariate normal

distribution over an offset ellipse. This function is tabulated in at least three places.[5, 6, 7] The notation was chosen to conform to that of Mr. Builte.[8] J. H. Cadwell,[9] has shown that

$$E(a', b', k', h') \approx \frac{2a'b'}{\left[\left(a'^2+4\right)\left(b'^2+4\right)\right]^{\frac{1}{2}}} \exp -2\left[\frac{h'^2}{a'^2+4}+\frac{h'^2}{b'^2+4}\right] \quad (29)$$

Mr. Cadwell showed that the error from using this approximation is less than .002 if a' is less than 0.5. He also gives approximations which can be used for a' less than 1.5 and for a' less than 2.5. However, with comparable accuracy these latter approximations are rather cumbersome.

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[5]Rodden, J. J. and G. W. Rosenthal, Tables of the Integral of the Elliptical Bivariate Normal Distribution Over Offset Circles, Lockheed LMSD - 900619, May 1, 1961.

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[6]DiDonato, A. R. and M. P. Jarnagin, Integration of the General Bivariate Gaussian Distribution Over an Offset Ellipse, Naval Weapons Laboratory Report NWL 1710, August 11, 1960.

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[7]Builte, A. G., A Building Block Technique for the Statement and Solutions of the Problems Involving the Bivariate Normal Distribution, Master's Thesis, School of Engineering, Air University, August 1962, p. 119, pp. 54-50.

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[8]Ibid.

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[9]Cadwell, J. H., "An Approximation to the Integral of the Circular Gaussian Distribution Over an Offset Ellipse", Mathematics of Computation, January 1964, Volume 18, No. 85, pp. 106-113.

In the special case where $a' = b'$, the circular coverage function, $p(a', r)$, of H. H. Germond is applicable.[10] In this case,

$$E(a', b', k', h') = E(a', a', k', h') = p(a', r)$$

where,

$$r'^2 = h'^2 + k'^2 \quad .$$

An extensive tabulation of the Q function where,

$$Q(a', r) = 1 - p(a', r)$$

exists and offers the best source of the values of the circular coverage function.[11]

Roger Snow showed that the circular coverage function, $p(a', r)$ can also be used when $a' \neq b'$ if $k' = h' = 0$.[12] In this case,

$$E(a', b', 0, 0) = q(a', b') = p\left(\frac{a'+b'}{2}, \frac{a'-b'}{2}\right) - p\left[\frac{(a'-b'}{2}, \frac{a'+b')}{2}\right]. \tag{30}$$

If both $a' = b'$ and $k' = h' = 0$, then $E(a', a', 0, 0) = 1 - e^{-\frac{1}{2}a'^2}$

i.e., $E(a', a', 0,0)$ is the integral of the well-known Rayleigh distribution. In this case the tables of e^{-x} can be used to evaluate the integral.[13] If the accuracy requirements are not

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[10] Germond, H. H., The Circular Coverage Function, Rand Corporation RM 330, January 26, 1950.

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[11] Marcum, J. I., Tables of Q Functions, Rand Corporation RM 339, January 1950.

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[12] Snow, Roger, Some Characteristics of the Elliptic Gaussian Distribution. Rand Corporation RM 2765-PR, September 1961.

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[13] National Bureau of Standards, Tables of the Exponential Function e^x, U. S. Government Printing Office. 1947.

to be stringent, special graph paper can be used to evaluate $E(a', a', 0, 0)$.[14]

If $k'(t')$ and $h'(t')$ or $a'(t')$ and $b'(t')$ are not constants the integral can be evaluated by the following procedure:

(1) For a given $t' = t_i'$ evaluate $k'(t_i')$ and $h'(t_i')$ or $a'(t_i')$ and $b'(t_i')$.

(2) Find $E[a'(t_i'), b'(t_i'), k', h']$ from one of the references given or from equation (29).

(3) From the normal probability tables find
$$f(t_i') = (\tfrac{1}{2\pi})^{1/2} e^{-\frac{1}{2} t_i'^2}$$

(4) From the product,
$$f(t_i') E(a_i', b_i', k_i', h_i')$$
This determines a point of the frequency function, $p(t')$.

(5) Repeating this procedure, with t' varying between the lower limit L_1' and upper limit U_1', determines the entire frequency function.

(6) Numerical integration of this frequency function yields the probability content P.

5. Example

As an example of the above procedure the probability content of the lethal volume, will now be computed for a special case of End Game I. In End Game I if $\ell_1 = 1$, and $\ell_2 = \ell_3 = 0$, then an antiparallel intercept exists. In this case $a = b = R_L$ and the cylinder is right circular. If the mean of the distribution is assumed to be located at the base of the interior cone. (see figure 10); then, $k' = h' = 0$, and equations (28A) and (30) hold.

[14] Burke, T. Finley, New Graph Paper for Circular Normal Distributions, Rand Corporation RM 3292-PR, September 1962.

To fix the dimensions of the lethal volume, let $|\overline{BC}| = R_L$ and $\frac{V_z}{V_R} = \frac{1}{2}$. To fix the distribution, let $\sigma_x = \sigma_y = \sigma_z = R_{L/3}$.

For the cylinder, then $U_{11}' = 3.0$ and $L_{11}' = 0.0$. Equations (28B) and (28C) then yield,

$$\Delta\Omega = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^{3.0} e^{-\frac{t^2}{2}} dt = 0.4987$$

From equation (30) is obtained

$$E = 1 - e^{-\frac{1}{2}(3)^2} = 0.98889 \quad .$$

Hence,

$$P_1 = \Delta\Omega \; E = 0.4940 \quad .$$

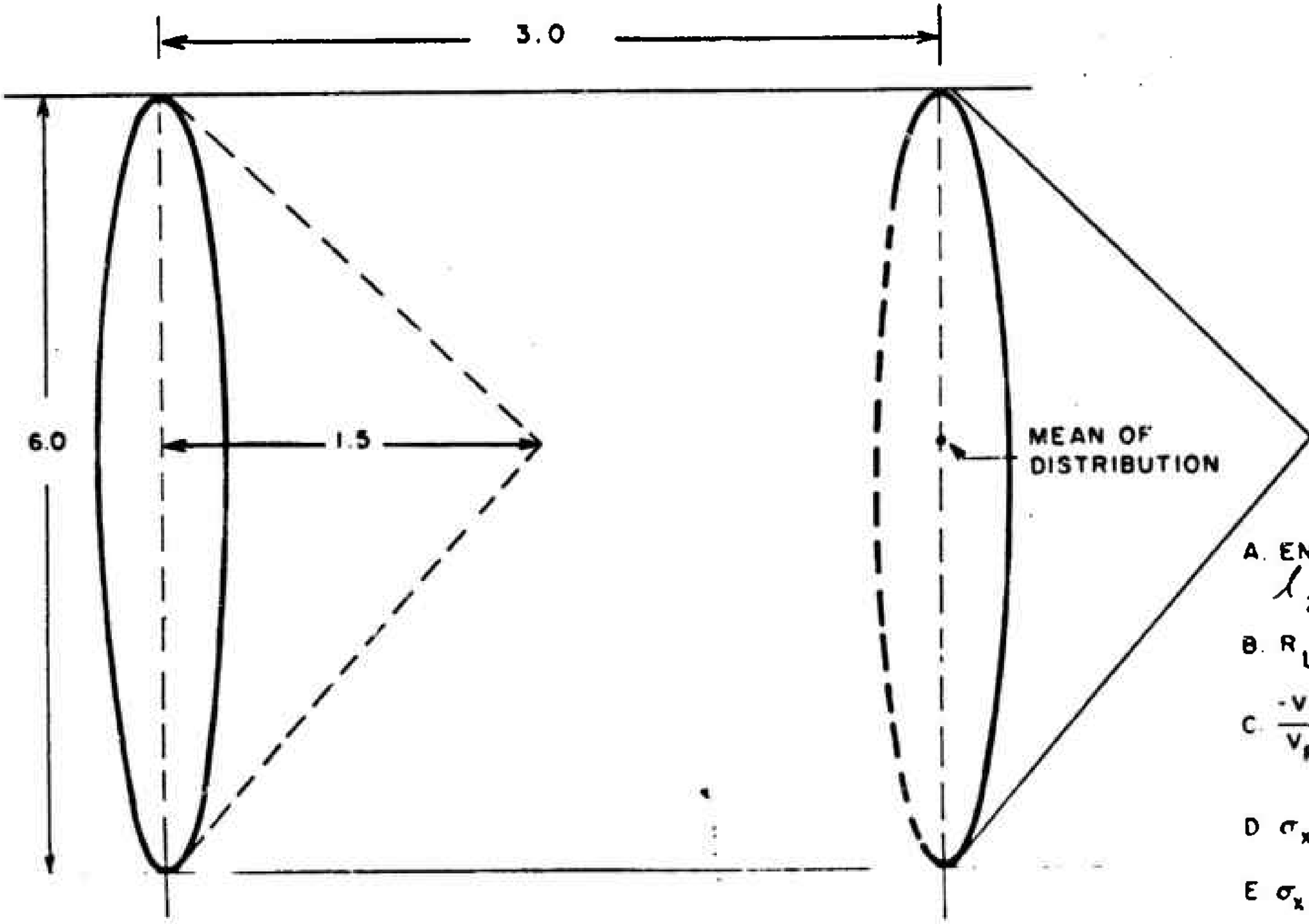
Next consider the problem of finding P_3. From figure 10 the cone exterior to the volume is seen to be oriented with its apex location at $t_z' = +1.5 = L_{21}$. Since $V_z/V_R = \frac{1}{2}$, $U_{21} = L_{21} + (\frac{1}{2}) R_L/\sigma_t = 3.0$.

Following the procedure outlined in the last section, Table 2 can now be generated.

Table 2. Calculation of P_3

t'	$a'(t')$	$1 - e^{-\frac{1}{2}a'(t')^2}$	$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t'^2}$	$p_3(t')$
1.5	0	0	.12952	0
1.6	0.2	.01980	.11092	.00220
1.7	0.4	.07688	.09405	.00723
•	•	•	•	•
•	•	•	•	•
•	•	•	•	•
3.0	3.0	.98887	.00443	.00438

Where $p_3(t') = \left[1 - e^{-\frac{1}{2}[a'(t')]^2}\right]\left(\frac{1}{2\pi} e^{-\frac{1}{2}t'^2}\right)$.



ASSUMPTIONS

A. ENDGAME I WITH $\ell_1 = -1$, $\ell_2 = \ell_3 = 0$

B. $R_L = |\overline{BC}|$

C. $\frac{-V_x}{V_R} = \frac{1}{2}$

D $\sigma_x = \sigma_y = \sigma_z$

E $\sigma_x = \frac{R_L}{3}$

Figure 10 NORMALIZED LETHAL VOLUME FOR ANTIPARALLEL INTERCEPT

Next $p_3(t')$ is plotted in figure 11. The area under this curve is estimated using one of numerous techniques available yielding,

$$P_3 = \int_{1.5}^{3.0} p_3(t')\,dt = 0.0188 \quad .$$

P_2 can be found by a similar procedure. From figure 10 the cone interior to the volume is seen to have its base nearest the mean of the distribution. For the interior cone the function $a'(t')$ must be a maximum, (3), at $t'=0$ and a minimum, (0), at $t'=-1.5$. The function $p_2(t')$ is plotted in figure 12. Then

$$P_2 = \int_{-1.5}^{0} p_2(t')\,dt = 0.285 \quad .$$

Now the probability content of the volume is given by

$$P_t = P_1 + P_2 - P_3 = 0.7602 \quad .$$

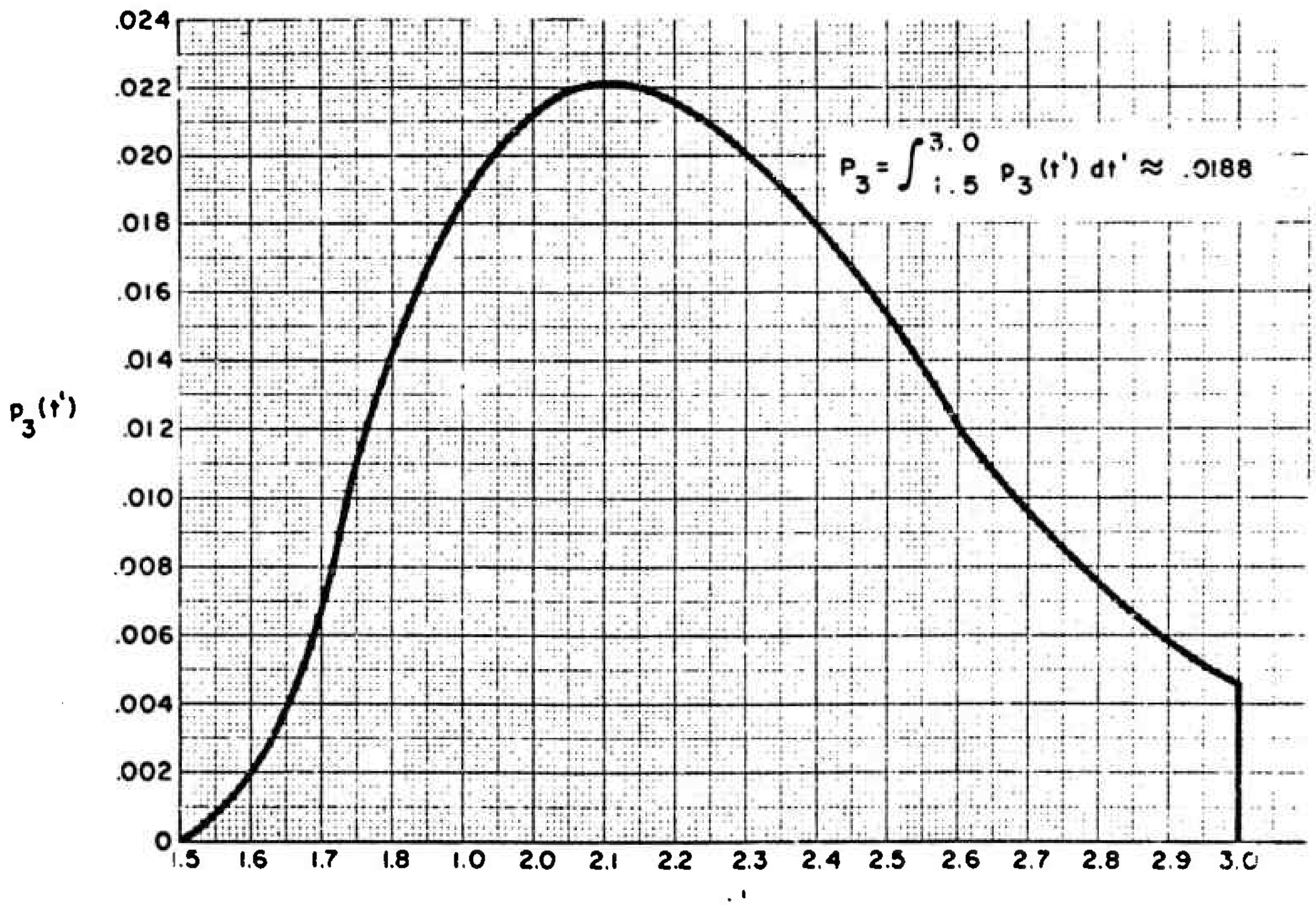


FIGURE 11. THE FREQUENCY FUNCTION USED FOR DETERMINING P_3

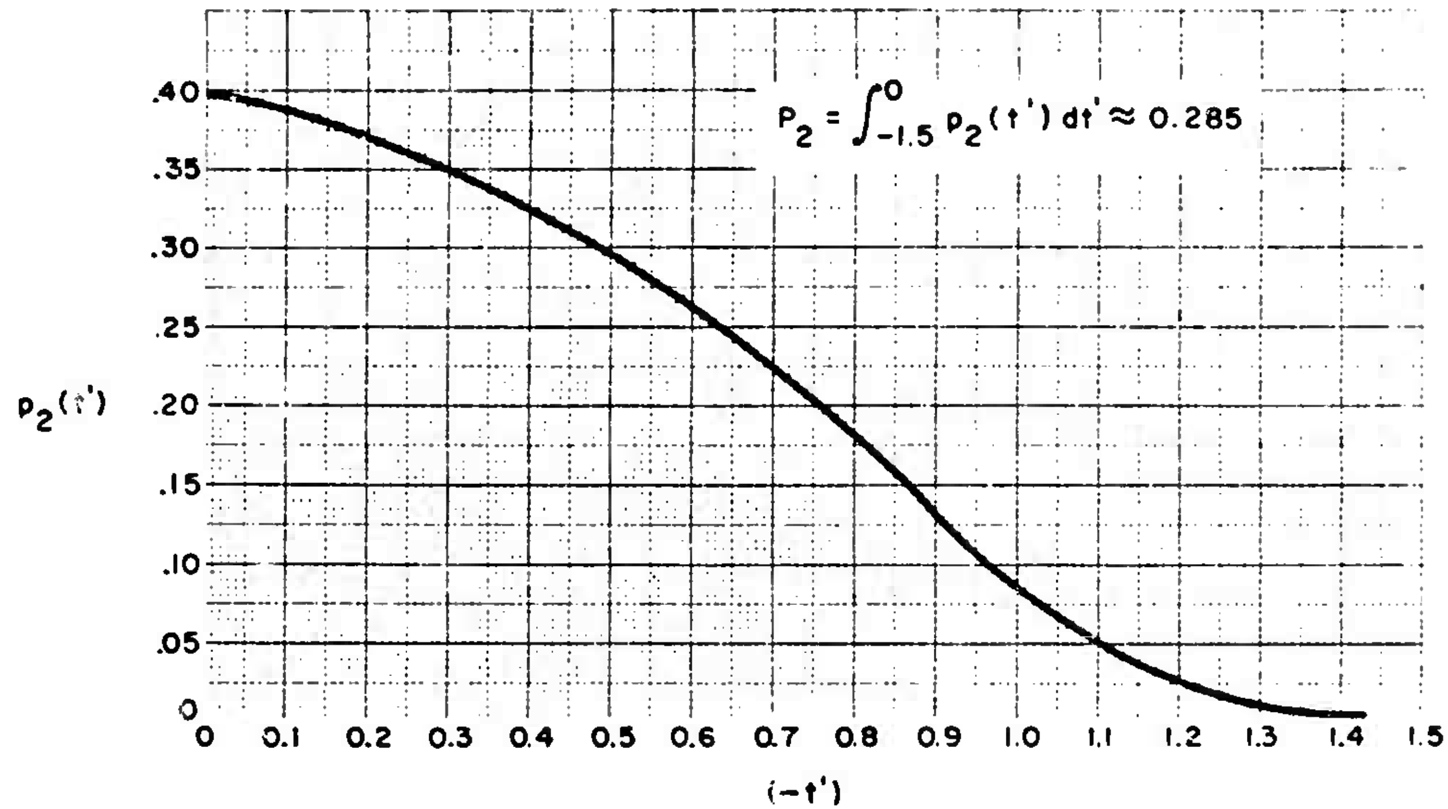


Figure 12. THE FREQUENCY FUNCTION USED FOR DETERMINING P_2

CHAPTER 4

PARAMETER VARIATION STUDIES

Using the results of Chapter 3, the variation in the size and the probability content of the lethal volume with the model parameters can be investigated. In a typical investigation all except one of the model inputs are held constant and the variation in the measures of effectiveness is expressed as a function of the remaining parameters. The values assigned to the inputs held constant are then changed and the variation in the measures of effectiveness is again expressed as a function the remaining parameter. This process is repeated until the change in the measure of effectiveness can be reliably predicted for any set of values assigned to the inputs held constant.

Another way in which the variation in the measures of effectiveness is sometimes investigated is the tradeoff study. For the tradeoff study all except two of the inputs are held constant. Combinations of these two inputs are then found such that the measures of effectiveness remain constant. Here, a popular example is a set of iso-kill probability curves. Having such curves the analyst can, if he has the cost data, select the combination of the two variables which give a satisfactorily high kill probability for the least cost.

The construction of iso-kill probability curves in this fashion may be extremely difficult unless the probability of kill function can be solved explicitly for the variables of interest. Since the kill probability function cannot usually be

solved explicitly, an iterative process must be used. This can be quite expensive if high accuracy is desired.

One way this problem can be surmounted is to compute the probability of kill as a function of the ratio of two of the model inputs. If the probability of kill can be computed in this fashion, then the tradeoffs between two of the inputs can still be noted and the calculations can still be easily performed.

A study in which the variation of kill probability is expressed as a function of the ratio of the input variables will not be illustrated. To obtain the ratios, the trivariate normal frequency function f(t, u, v) is restricted to the spherical normal frequency function; i.e.,

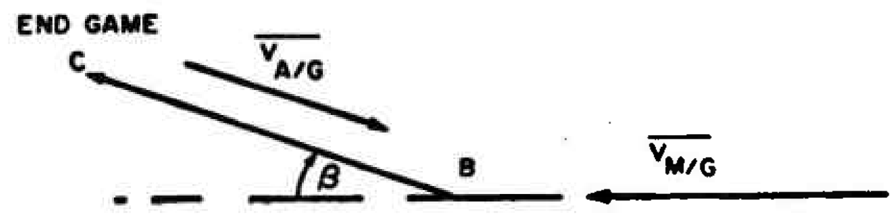
$$\sigma_t = \sigma_u = \sigma_v = \sigma \quad .$$

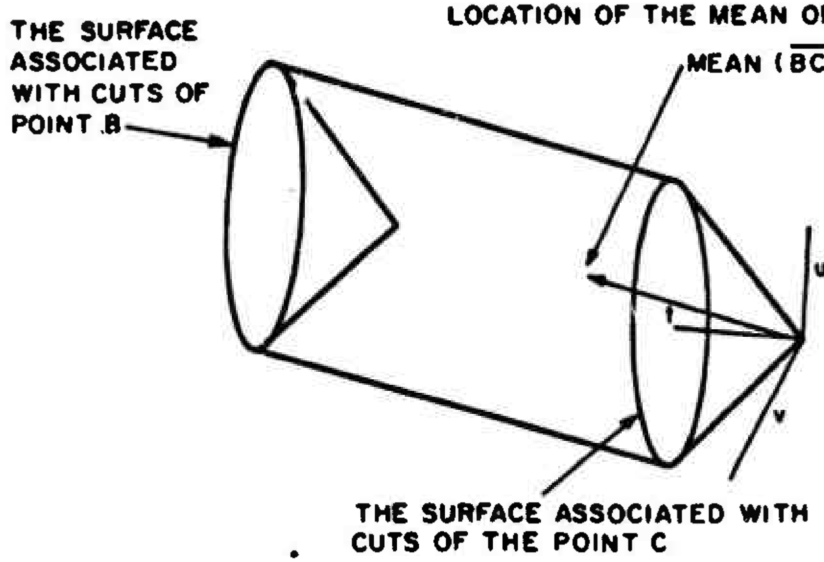
The probability content of the lethal volume can now be computed as a function of the ratio of the maximum opening radius of the rod, R_L, to or as a function of the ratio of the effective length of the member $|\overline{BC}|$, to σ.

Figure 13 shows the location of the mean of the frequency function taken, the values assigned for the other inputs, and the range of the ratios that were investigated.

The results of this investigation are shown in figures 14 and 15. Figure 14 shows the size of the lethal volume normalized with respect to σ^3 as a function of the ratio R_L/σ. Figure 15 shows the variation in the probability content of the lethal volume for the same inputs.

The lethal volume model can be similarly used to determine the vsriation in kill probability with any of the parameters which





OTHER ASSUMPTIONS

1. $|\overline{V_{A/G}}| = |\overline{V_{M/G}}| = 1/4\,|V_R|$
2. $|\overline{BC}| = 2\sigma$ OR 4σ
3. $R_{L/\sigma} = 1-5$
4. $\beta = 0°, 30°, 60°$

Figure 13. ASSUMPTIONS FOR FIGURES 14 AND 15

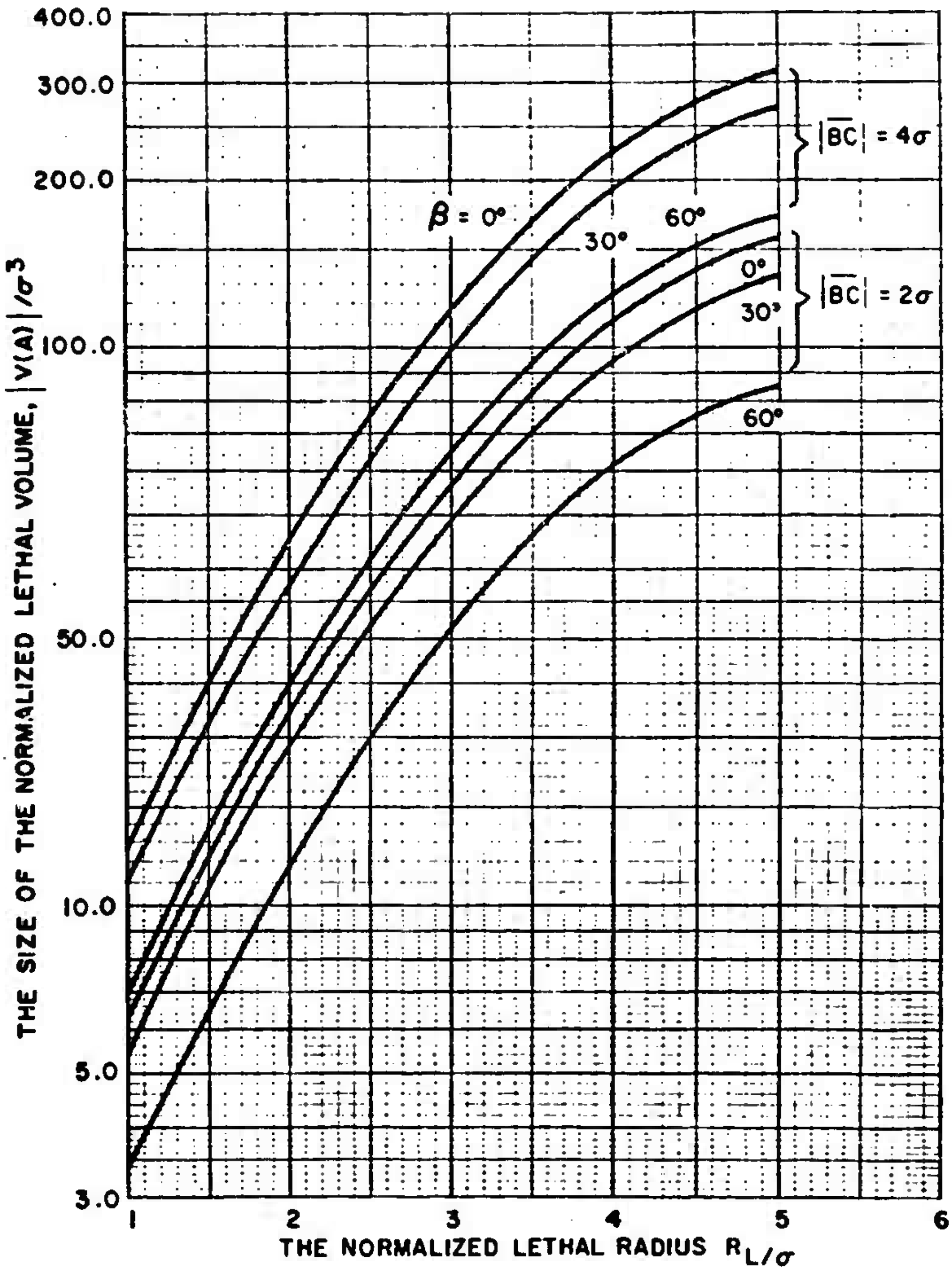


Figure - 14. THE VARIATION IN THE SIZE OF THE LETHAL VOLUME WITH THE MODEL INPUTS

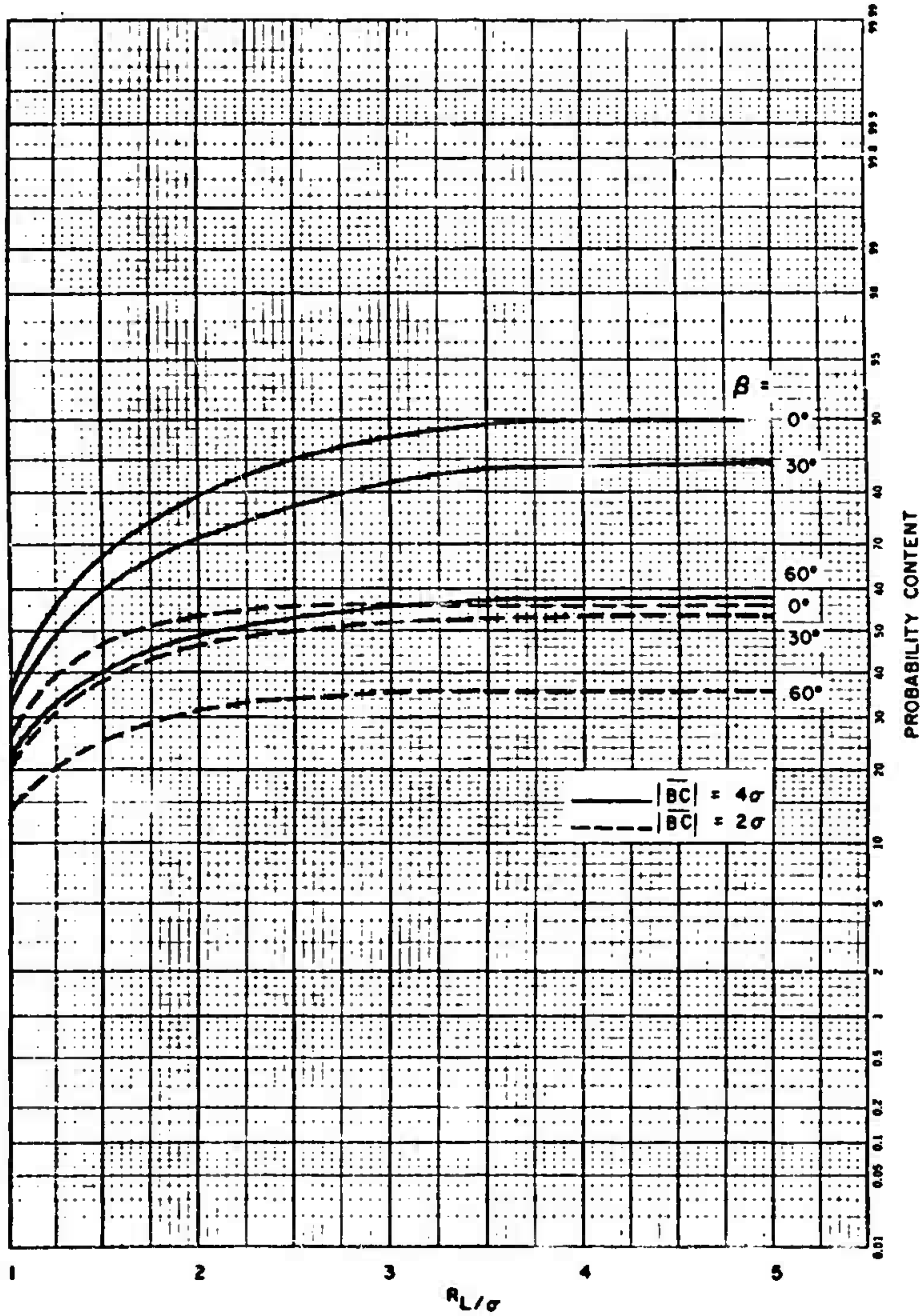


Figure 15. VARIATION IN THE PROBABILITY CONTENT OF THE LETHAL VOLUME WITH MODEL INPUTS

are inputs to the model. The number of parameters of the model can be increased by deriving the relationship between the inputs and other end game variables. For example, the standard deviation of frequency distribution can be divided into guidance and timing errors. Parameter variation studies can then be conducted with either the guidance or timing errors. A further division such as thia increases the resolution of the model. Table 3 lists some of the model inputs together with ways these inputs might be divided to increase the number of parameter variation studies possible with the model.

Table 3
Division of the Model Inputs

1. Standard deviation of the frequency distribution
   a. Guidance errors
   b. Timing errora
   c. Target maneuvers
2. Lethal radius of the rod
   a. Warhead weight
   b. Warhead size and shape
3. Average rod velocity
   a. Warhead weight
   b. Warhead sizc and shape
   c. Engagement altitude
4. Effective length of target
   a. Target shape
   b. Target hardness
   c. Closing velocity

5. Velocity vector of the missile

   a. Range to intercept

   b. Angle between the missile's axis and its velocity vector

CHAPTER 5

MODEL IMPROVEMENTS

INTRODUCTION

In this final section, a means of improving the target and warhead models will be presented. A model improvement is defined as a more realistic mathematical description of the actual performance of the hardware during intercept.

The need for model improvements depends on the particular variables to be investigated, the purpose of the study, the accuracy required, and the reliability of the inputs to the model. For example, if the object of a study is to accurately estimate the kill probability of a system, then a more realistic description of the target and warhead would be desirable. On the other hand, if the object of the study is to roughly estimate the gain in kill probability obtained by improving the guidance accuracy, then the current model will suffice.

TARGET MODEL IMPROVEMENTS

The lethal volume associated with cuts of a structural member has been derived. The lethal volume associated with cuts of a stick aircraft can now be determined by taking the union of the lethal volume associated with each of the structural members of an aircraft. One of the ways that this may be accomplished is given below.

Consider the aircraft sketched in figure 16. If the point where the wings are attached to the fuselage is selected as the origin, then the target can be described with four vectors. One

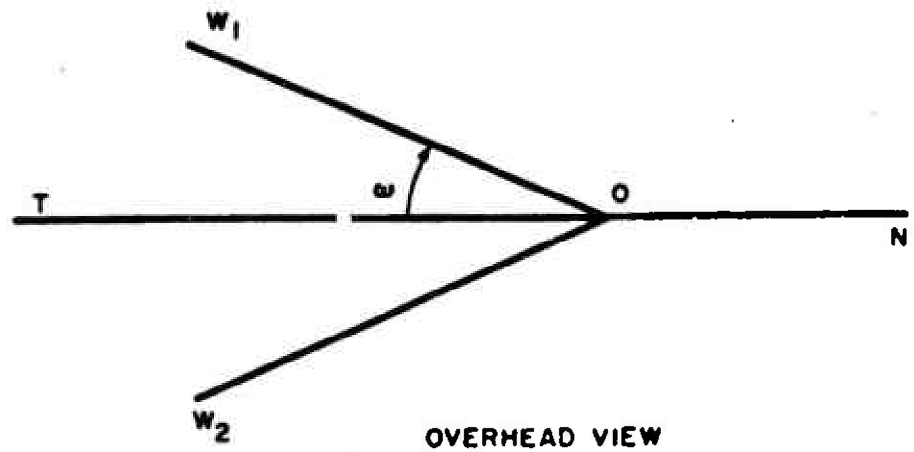


Figure 16. STICK AIRCRAFT TARGET MODEL

vector, $\overline{N}$, has a magnitude equal to the length from 0 to N, and is directed from 0 to N. Similarly, vectors from $\overline{T}$, $\overline{W_1}$, and $\overline{W_2}$ are defined as the vectors from 0 to T, W_1, and W_2 respectively.

The total lethal volume V_L, associated with cuts of all four vectors is then given by, $V_L = V_N \cup V_T \cup V_{W_1} \cup V_{W_2}$

Figure 17 gives the cross-section of V_L in the xy plane for an antiparallel intercept (End Game I with $\ell_1 = \pm 1$). This approach ignores the cumulative effects which might occur when more than one structural member is hit; i.e., when two members are both subjected to a sub-lethal effect, their cumulative effect may well be lethal to the aircraft. The changes of such an occurrence are remote with a continuous rod warhead.

The major difficulty that arises in making this model improvement is in expressing the limits for the integral,

$$P_k = \iiint_{V_L} f(t, u, v)\ dtuddv \quad .$$

Ideally the limits of integration could be expressed as functions of the wing length $|\overline{W_1}|$, the wing angle ω, and the roll of the aircraft. Such a derivation would be an extremely difficult but certainly a feasible undertaking.

To better estimate the desirability of undertaking the target model improvement, an estimate of the change in kill probability due to adding the wings to the target is needed. If the vector $\overline{NT}$ is considered to represent the segment $|\overline{BC}|$ for the single linear structural member model already developed, then the consequences of adding the vectors $\overline{W_1}$ and $\overline{W_2}$ to the model can be considered.

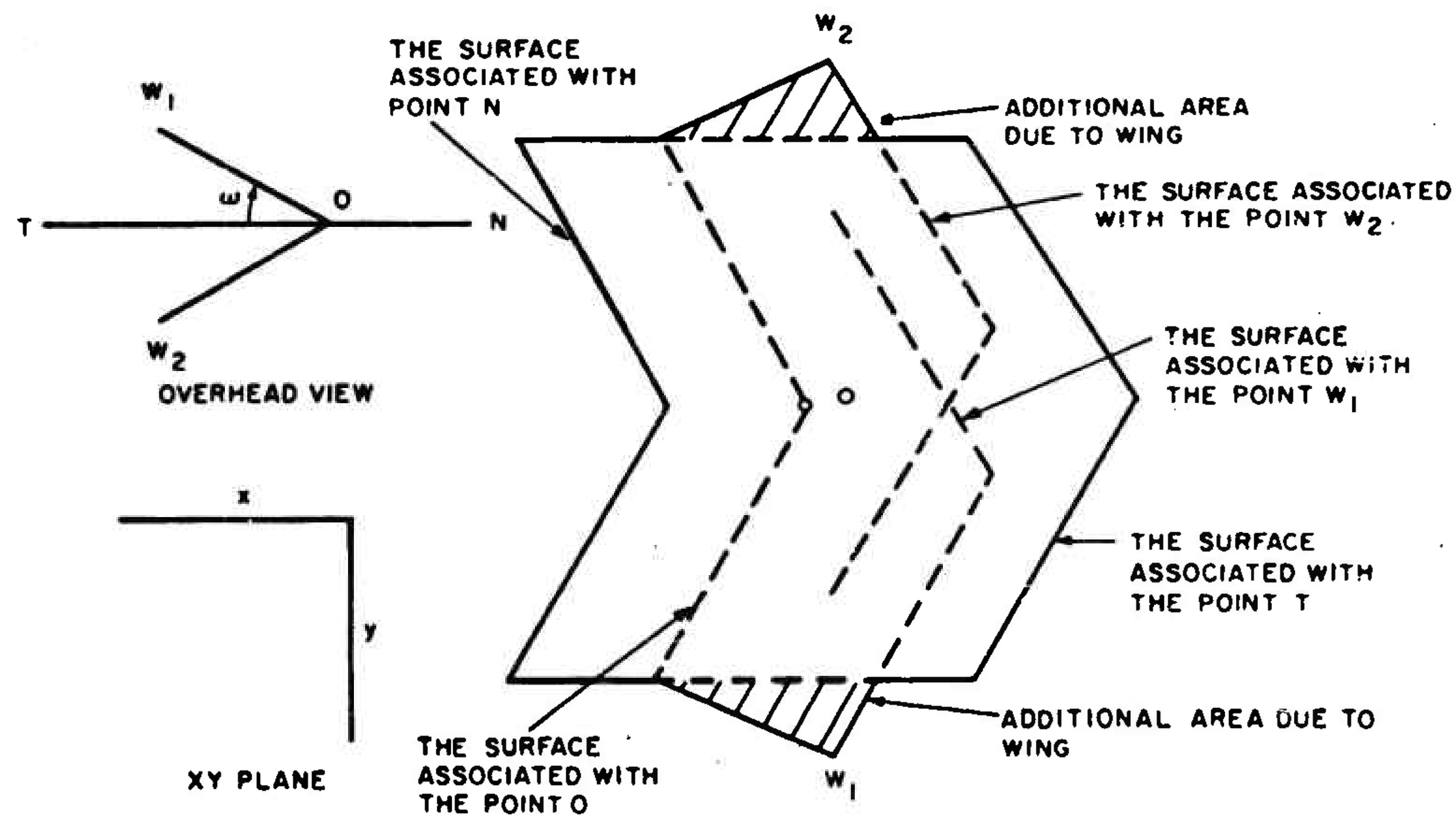


Figure 17. CROSS SECTION OF THE LETHAL VOLUME FOR AN ANTIPARALLEL INTERCEPT OF AN AIRCRAFT

From equation (31) it is seen that since

$$V_L = V_{NT} \cup (V_{W_1} \cup V_{W_2}) \geq V_F$$

the probability content with the stick aircraft model will in general, be greater than that obtained by using the single member model. In figure 17, the cross section of $V_L - (V_{W_1} \cup V_{W_2})$ has been shaded. The probability content of $V_L - (V_{W_1} \cup V_{W_2})$ represents the increase in probability due to the addition of the wings. For a particular end game the probability content of $V_L - (V_{W_1} \cup V_{W_2})$ can be bounded in numerous ways, e.g., using Gaussian equal cell probability paper. From this increase in probability the analyst can judge the importance of the target model improvement for his particular application.

WARHEAD MODEL IMPROVEMENTS

One of the principal assumptions made before the expressions for the size and probability content of the lethal volume are derived in Section D, was that the rod travels with a constant velocity. In the sequel, a means of accounting for a variable rod velocity and the effects of this model improvement will be discussed.

If an object is ejected into the atmosphere with high initial velocity, its velocity will decay rapidly as a function of the distance traveled. This velocity will approximate the equation,

$$V(R) = V_o \, e^{-KR} \tag{32}$$

where,

$V(R)$ is velocity of the rod at the distance R,

$V(o)$ is the ejection velocity of the rod,

and K is the decay parameter.

The parameter K is a function of the drag encountered; hence, K varies with the ballistics of the rod and with the air density. The parameter K, then will not only change with the characteristics of the rod, but will also change with the altitude of the burst.

The time required for the rod to trsvel a distance R is given by,

$$t_R = \int_0^R \frac{dR}{V(R)} = \frac{1}{K\,V(0)}\left(e^{KR}-1\right)$$

The average velocity of the rod to a distance R, the parameter V_R used in Chapter II, is given by,

$$V_R = \frac{R}{t_R} = \frac{K\,R\,V(0)}{e^{KR}-1}$$

When $V_y = V_z = 0$, equation (13) gives

$$S(B) = \frac{z_B^2}{R^2} + \frac{y_B^2}{R^2} - \frac{x_B^2}{\left(\frac{R\,V_x}{V_R}\right)^2}$$

Where $S(B) = 0$ is the equation of the surface associated with cuts of the point B.

Substituting for V_R gives,

$$S'(B) = \frac{z_B^2}{R^2} + \frac{y_B^2}{R^2} - \frac{x_B^2}{\left[\frac{K\,V_o\,V_x\,R^2}{e^{KR}-1}\right]^2}$$

Following a procedure similar to that unsed in Chapter 3, the lethal volume can be formed by connecting the surface defined by $S'(B) = 0$ to a similar surface defined by $S'(C) = 0$ with parallel line segments. Then the expressions for the size and probability content of the lethal volume can be developed following

the techniques of Chapter 4. Again the major difficulty would be in expressing the limits of the integration to obtain the probability content.

Note that since K is greater than zero the surface defined $S'(B)=0$ contains a larger volume than the surface defined by $S(B) = 0$. (See figure 18). Again an estimate of the change in kill probability due to the model improvement is desired.

If P_C is the probability content of the surface defined by $S(B) = 0$ (obtained with $V_R = V_{R_L}$) and P_C' is the probability content of the surface defined by $S'(B) = 0$ (using equation (32) for V_R) then the problem is to estimate

$$P_C' - P_C \quad .$$

Let the axis of the surface defined by $S'(B) = 0$ be divided into n intervals and let $j = 1, 2, -- n$, be the coordinate along the axis at the end of the n^{th} interval. Let R_j' denote the distance from the axis to the surface defined by $S'(B)$ at the point j. Let the probability content of the cones of radius R_j' obtained with $V_R = V_{R_j}$ be P_j. Let R_j be the distance from the j^{th} coordinate to the cone $S(B) = 0$ and let P_j be the probability content of this cone.

Note that $P_s' = P_s = P_C$. Then,

$$P_C' - P_C = \lim_{n \to \infty} \sum_{j=1}^{n} (P_j' - P_j) \tag{33}$$

taking a small n will yield the desired approximation.

A similar procedure will yield P_1', P_2' and P_3' corresponding to the probability content of the cones and the cylinder. The same procedure can also be used when either V_y and V_z are not both zero. The analyst can again judge whether or not this model

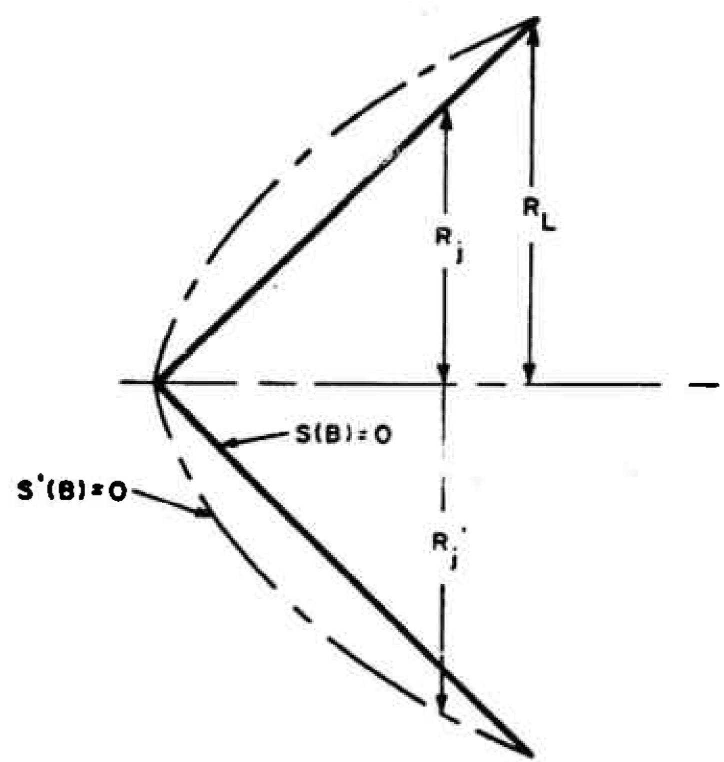


Figure 18. CROSS SECTION OF THE SURFACES DEFINED BY S(B) AND S(B)' = 0

improvement is worthwhile.

Incidentally, equation (33) offers an alternate means of obtaining the probability content with variable rod velocity. Using equation (33) to obtain a high degree of accuracy would be rather expensive, however.

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